

# QUIVER GRASSMANNIANS, QUIVER VARIETIES AND THE PREPROJECTIVE ALGEBRA

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**ABSTRACT.** Quivers play an important role in the representation theory of algebras, with a key ingredient being the path algebra and the preprojective algebra. Quiver grassmannians are varieties of submodules of a fixed module of the path or preprojective algebra. In the current paper, we study these objects in detail. We show that the quiver grassmannians corresponding to submodules of certain injective modules are homeomorphic to the lagrangian quiver varieties of Nakajima which have been well studied in the context of geometric representation theory. We then refine this result by finding quiver grassmannians which are homeomorphic to the Demazure quiver varieties introduced by the first author, and others which are homeomorphic to the graded/cyclic quiver varieties defined by Nakajima. The Demazure quiver grassmannians allow us to describe injective objects in the category of locally nilpotent modules of the preprojective algebra. We conclude by relating our construction to a similar one of Lusztig using projectives in place of injectives.

## CONTENTS

Introduction	1
1. Quivers, path algebras, and preprojective algebras	3
2. Modules of the path algebra and quiver grassmannians	4
3. Quiver varieties	8
4. Relation between quiver grassmannians and quiver varieties	10
5. Group actions and graded quiver grassmannians	14
6. Geometric construction of representations of Kac-Moody algebras	18
7. Relation to Lusztig's grassmannian realization	22
References	26

## INTRODUCTION

Quivers play a fundamental role in the theory of associative algebras and their representations. Gabriel's theorem, which states a precise relationship between indecomposable representations of certain quivers and root systems of associated Lie algebras, indicated that the representation theory of quivers was also intimately connected to the representation theory of

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Kac-Moody algebras. This eventually lead to the Ringel-Hall construction of quantum groups and the quiver variety constructions of Lusztig and Nakajima.

Fix a quiver (directed graph)  $Q = (Q_0, Q_1)$  with vertex set  $Q_0$  and arrow set  $Q_1$ . The corresponding path algebra  $\mathbb{C}Q$  is the algebra spanned by the set of directed paths, with multiplication given by concatenation. There is a natural grading  $\mathbb{C}Q = \bigoplus_n (\mathbb{C}Q)_n$  of the path algebra by length of paths. Representations of a quiver are equivalent to representations (or modules) of its path algebra. Note that  $(\mathbb{C}Q)_0$ -modules are simply  $Q_0$ -graded vector spaces, and in particular all  $\mathbb{C}Q$ -modules are  $Q_0$ -graded. For a  $\mathbb{C}Q$ -module  $V$  and  $u \in \mathbb{N}Q_0$ , the associated *quiver grassmannian* is the variety  $\text{Gr}_Q(u, V)$  of all  $\mathbb{C}Q$ -submodules of  $V$  of graded dimension  $u$ . These natural objects (or closely related ones) can be found in several places in the literature. For instance, they appear in [6, 31] in the study of spaces of morphisms of  $\mathbb{C}Q$ -modules and in [3, 4, 10] in connection with the theory of cluster algebras. Geometric properties have also been studied in [5, 32, 33].

Let  $\mathfrak{g}$  be the Kac-Moody algebra whose Dynkin diagram is the underlying graph of  $Q$  (the graph obtained by forgetting the orientation of all arrows) and let  $\tilde{Q}$  be the *double quiver* obtained from  $Q$  by adding an oppositely oriented arrow  $\bar{a}$  for every  $a \in Q_1$ . One is often interested in modules of the preprojective algebra  $\mathcal{P} = \mathcal{P}(Q)$ , which is a certain natural quotient of the path algebra  $\mathbb{C}\tilde{Q}$  and inherits the grading. In particular,  $\mathcal{P}$ -modules are also  $\mathbb{C}\tilde{Q}$ -modules. To each vertex  $i \in Q_0$ , we have an associated one-dimensional simple  $\mathcal{P}$ -module  $s^i$ . For  $w = \sum_i w_i i \in \mathbb{N}Q_0$ , we let  $s^w = \bigoplus_i (s^i)^{\oplus w_i}$  be the corresponding semisimple module. By the Eckmann-Schöpf Theorem the category of  $\mathcal{P}$ -modules has enough injectives, so we can define  $q^w$  to be the injective hull of  $s^w$ . One of the main results of the current paper is that the quiver grassmannian  $\text{Gr}_{\tilde{Q}}(v, q^w)$  is homeomorphic to the lagrangian Nakajima quiver variety  $\mathfrak{L}(v, w)$  used to give a geometric realization of irreducible highest weight representations of  $\mathfrak{g}$  (see [20, 21]). Furthermore, for each  $\sigma$  in the Weyl group of  $\mathfrak{g}$ , there is a natural finite-dimensional submodule  $q^{w, \sigma}$  of  $q^w$  such that the quiver grassmannian  $\text{Gr}_{\tilde{Q}}(v, q^{w, \sigma})$  is homeomorphic to the Demazure quiver variety  $\mathfrak{L}_\sigma(v, w)$  defined by the first author [30]. One benefit of this realization of the quiver varieties is that it avoids the description as a moduli space. One can view it as a uniform way of picking a representative from each orbit in the original moduli space description.

Quiver grassmannians admit natural group actions. We describe these actions and show that certain special cases agree, under the homeomorphisms described above, with well-studied groups actions on Nakajima quiver varieties. In this way, we are able to give a quiver grassmannian realization of the cyclic/graded quiver varieties used by Nakajima to define  $t$ -analogs of  $q$ -characters of quantum affine algebras [23].

The injective modules  $q^w$  are locally nilpotent if and only if the quiver  $Q$  is of finite or affine type. However, it turns out that the submodules  $q^{w, \sigma}$  are always nilpotent. The limit  $\tilde{q}^w$  of these submodules is the injective hull of the semisimple module  $s^w$  in the category of locally nilpotent  $\mathcal{P}$ -modules, giving us a description of the indecomposable injectives in this category.

Lusztig has previously presented a canonical bijection between the points of the lagrangian Nakajima quiver variety and the points of a type of quiver grassmannian inside a projective (as opposed to injective) object. In finite type, the projective objects are also injective. It turns out that, on the level of geometric realizations of representations of finite type  $\mathfrak{g}$ , the two constructions are related by the Chevalley involution.

Throughout the current paper, we work over the field  $\mathbb{C}$  of complex numbers. While many results hold in more generality, this assumption will streamline the exposition and several results we quote in the literature are stated over  $\mathbb{C}$ . We will always use the Zariski topology and do not assume that algebraic varieties are irreducible. We let  $\mathbb{N} = \mathbb{Z}_{\geq 0}$  and denote the fundamental weights and simple roots of a Kac-Moody algebra by  $\omega_i$  and  $\alpha_i$  respectively.

This paper is organized as follows. In Section 1 we review some results on quivers, path algebras and preprojective algebras. In Section 2 we discuss various module categories of these objects and introduce our main object of study, the quiver grassmannian. We review the definition of the quiver varieties of Lusztig and Nakajima in Section 3 and realize these as quiver grassmannians in Section 4. In Section 5 we introduce a natural group action and show how it can be used to recover group actions typically constructed on quiver varieties. In Section 6 we use quiver grassmannians to give a geometric realization of integrable highest weight representations of a symmetric Kac-Moody algebra. Finally, in Section 7 we discuss a precise relationship between our construction and a similar one due to Lusztig.

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## 1. QUIVERS, PATH ALGEBRAS, AND PREPROJECTIVE ALGEBRAS

In this section we briefly review the relevant definitions concerning quivers. We refer the reader to [9, 25, 27] for further details.

A *quiver* is a directed graph. That is, it is a quadruple  $Q = (Q_0, Q_1, s, t)$  where  $Q_0$  and  $Q_1$  are sets and  $s$  and  $t$  are maps from  $Q_1$  to  $Q_0$ . We call  $Q_0$  and  $Q_1$  the sets of *vertices* and *directed edges* (or *arrows*) respectively. For an arrow  $a \in Q_1$ , we call  $s(a)$  the *source* of  $a$  and  $t(a)$  the *target* of  $a$ . Usually we will write  $Q = (Q_0, Q_1)$ , leaving the maps  $s$  and  $t$  implied. The quiver  $Q$  is said to be *finite* if  $Q_0$  and  $Q_1$  are finite. A *loop* is an arrow  $a$  with  $s(a) = t(a)$ . In this paper, all quivers will be assumed to be finite and without loops. A quiver is said to be of *finite type* if the underlying graph of  $Q$  (i.e the graph obtain from  $Q$  by forgetting the orientation of the edges) is a Dynkin diagram of finite *ADE* type. Similarly, it is of *affine* (or *tame*) *type* if the underlying graph is a Dynkin diagram of affine type and of *indefinite* (or *wild*) *type* if the underlying graph is a Dynkin diagram of indefinite type.

A *path* in  $Q$  is a sequence  $\beta = a_l a_{l-1} \cdots a_1$  of arrows such that  $t(a_i) = s(a_{i+1})$  for  $1 \leq i \leq l-1$ . We call  $l$  the *length* of the path. We let  $s(\beta) = s(a_1)$  and  $t(\beta) = t(a_l)$  denote the initial and final vertices of the path  $\beta$ . For each vertex  $i \in I$ , we have a trivial path  $e_i$  with  $s(e_i) = t(e_i) = i$ .

The *path algebra*  $\mathbb{C}Q$  associated to a quiver  $Q$  is the  $\mathbb{C}$ -algebra whose underlying vector space has basis the set of paths in  $Q$ , and with the product of paths given by concatenation. More precisely, if  $\beta = a_l \cdots a_1$  and  $\beta' = b_m \cdots b_1$  are two paths in  $Q$ , then  $\beta\beta' = a_l \cdots a_1 b_m \cdots b_1$  if  $t(\beta') = s(\beta)$  and  $\beta\beta' = 0$  otherwise. This multiplication is associative. There is a natural grading  $\mathbb{C}Q = \bigoplus_{n \geq 0} (\mathbb{C}Q)_n$  where  $(\mathbb{C}Q)_n$  is the span of the paths of length  $n$ .

Given a quiver  $Q = (Q_0, Q_1)$ , we define the *double quiver* associated to  $Q$  to be the quiver  $\tilde{Q} = (Q_0, \tilde{Q}_1)$  where

$$\tilde{Q}_1 = \bigcup_{a \in Q_1} \{a, \bar{a}\}, \quad \text{where} \quad s(\bar{a}) = t(a), \quad t(\bar{a}) = s(a).$$

We then have a natural involution  $\tilde{Q}_1 \rightarrow \tilde{Q}_1$  given by  $a \mapsto \bar{a}$  (where  $\bar{\bar{a}} = a$ ). The algebra

$$\mathcal{P} = \mathcal{P}(Q) = \mathbb{C}\tilde{Q} / \sum_{a \in Q_1} (a\bar{a} - \bar{a}a)$$

is called the *preprojective algebra* associated to  $Q$ . It inherits a grading  $\mathcal{P} = \bigoplus_{n \geq 0} \mathcal{P}_n$  from the grading on  $\mathbb{C}Q$ . Up to isomorphism the preprojective algebra  $\mathcal{P}(Q)$  depends only on the underlying graph of  $Q$ . See [16, §12.15] for details.

## 2. MODULES OF THE PATH ALGEBRA AND QUIVER GRASSMANNIANS

**2.1. Module categories.** For an associative algebra  $A$ , let  $A\text{-Mod}$  denote the category of  $A$ -modules and  $A\text{-mod}$  the category of finite-dimensional  $A$ -modules. We will use the notation  $V \in A\text{-Mod}$  (resp.  $V \in A\text{-mod}$ ) to indicate that  $V$  is an object in the category  $A\text{-Mod}$  (resp.  $A\text{-mod}$ ). Note that  $\mathcal{P}_0\text{-mod}$  is equivalent to the category of finite-dimensional  $Q_0$ -graded vector spaces whose morphisms are linear maps preserving the grading, and we will often blur the distinction between these two categories. Up to isomorphism, the objects of  $\mathcal{P}_0\text{-mod}$  are classified by their graded dimension. We denote the graded dimension of a module  $V$  by  $\dim_{Q_0} V = \sum_i (\dim V_i) i \in \mathbb{N}Q_0$  and let  $\dim_{\mathbb{C}} V = \sum_{i \in Q_0} \dim V_i \in \mathbb{N}$ . We will sometimes view the graded dimension  $\dim_{Q_0} V$  of  $V$  as its isomorphism class.

For  $V, W \in \mathcal{P}_0\text{-mod}$ , we denote the set of  $\mathcal{P}_0$ -module morphisms from  $V$  to  $W$  by  $\text{Hom}_{\mathcal{P}_0}(V, W)$ . Under the equivalence of categories above,  $\text{Hom}_{\mathcal{P}_0}(V, W)$  is identified with  $\bigoplus_{i \in Q_0} \text{Hom}_{\mathbb{C}}(V_i, W_i)$ . We define  $\text{End}_{\mathcal{P}_0} V$  to be  $\text{Hom}_{\mathcal{P}_0}(V, V)$  and  $\text{Aut}_{\mathcal{P}_0} V = \prod_{i \in Q_0} GL(V_i)$  to be group of invertible elements of  $\text{End}_{\mathcal{P}_0} V$ . For  $V \in \mathcal{P}_0\text{-mod}$ , we will write  $U \subseteq V$  to mean that  $U$  is an  $\mathcal{P}_0$ -submodule of  $V$ . This is the same as a  $Q_0$ -graded subspace. Note that any  $\mathcal{P}$ -module becomes a  $\mathcal{P}_0$ -module by restriction, and thus can be thought of as a  $Q_0$ -graded vector space.

Suppose  $A = \bigoplus_{n \geq 0} A_n$  is a graded algebra and  $V$  is an  $A$ -module. Then  $V$  is *nilpotent* if there exists an  $n \in \mathbb{N}$  such that  $A_n \cdot V = 0$ . We say  $V$  is *locally nilpotent* if for all  $v \in V$ , there exists  $n \in \mathbb{N}$  such that  $A_n \cdot v = 0$ . We denote by  $A\text{-lnMod}$  the category of locally nilpotent  $A$ -modules. For  $n \geq 0$ , we define  $A_{\geq n} = \bigoplus_{k \geq n} A_k$  and we let  $A_+ = A_{\geq 1}$ .

**Proposition 2.1.** *For a quiver  $Q$ , the following are equivalent:*

- (i)  $\mathcal{P}(Q)$  is finite-dimensional,
- (ii) all finite-dimensional  $\mathcal{P}(Q)$ -modules are nilpotent,
- (iii) all finite-dimensional  $\mathcal{P}(Q)$ -modules are locally nilpotent, and
- (iv)  $Q$  is of finite type.

*Proof.* The equivalence of (i) and (iv) is well-known (see for example [24]). That (ii) implies (iv) was proven by Crawley-Boevey [8] and the converse was proven by Lusztig [16, Proposition 14.2]. Since a finite-dimensional module is nilpotent if and only if it is locally nilpotent, (ii) is equivalent to (iii).  $\square$

**2.2. Simple objects.** For each  $i \in Q_0$ , let  $s^i$  be the simple  $\mathbb{C}\tilde{Q}$ -module given by  $s^i_i = \mathbb{C}$  and  $s^i_j = 0$  for  $i \neq j$ . Then  $s^i$  is also naturally a  $\mathcal{P}$ -module which we also denote by  $s^i$ .

**Lemma 2.2.** *The set  $\{s^i\}_{i \in I}$  is a set of representatives of the simple objects of  $\mathbb{C}\tilde{Q}\text{-lnMod}$  and  $\mathcal{P}\text{-lnMod}$ . In particular, if  $Q$  is of finite type, then  $\{s^i\}_{i \in I}$  is a set of representatives of the simple objects of  $\mathbb{C}\tilde{Q}\text{-Mod}$  and  $\mathcal{P}\text{-Mod}$ .*

*Proof.* Any nonzero element of a simple locally nilpotent module  $M$  generates a finite-dimensional module which must be all of  $M$ . Therefore  $M$  is finite-dimensional and hence nilpotent. Then  $(\mathbb{C}\tilde{Q})_+$  and  $\mathcal{P}_+$  are two-sided ideals of  $\mathbb{C}\tilde{Q}$  and  $\mathcal{P}$  respectively that act nilpotently on any nilpotent module. Therefore, simple nilpotent  $\mathbb{C}\tilde{Q}$ -modules and  $\mathcal{P}$ -modules are the same as simple  $\mathbb{C}\tilde{Q}/(\mathbb{C}\tilde{Q})_+$ -modules and  $\mathcal{P}/\mathcal{P}_+$ -modules respectively. Since

$$\mathbb{C}\tilde{Q}/(\mathbb{C}\tilde{Q})_+ \cong \mathcal{P}/\mathcal{P}_+ \cong \bigoplus_{i \in I} e_i,$$

the first statement follows. The second statement then follows from Proposition 2.1.  $\square$

**Lemma 2.3.** *Fix a quiver  $Q$  and let  $A$  be either  $\mathbb{C}\tilde{Q}$  or  $\mathcal{P}(Q)$ . If  $V \in A\text{-lnMod}$ , then the socle of  $V$  is  $\{v \in V \mid A_+ \cdot v = 0\}$ .*

*Proof.* It is clear that  $\{v \in V \mid A_+ \cdot v = 0\}$  is a sum of simple subrepresentations of  $V$  and is thus contained in the socle of  $V$ . Similarly, by Lemma 2.2, any simple subrepresentation of  $(V, x)$  is contained in  $\{v \in V \mid A_+ \cdot v = 0\}$ . The result follows.  $\square$

**2.3. Projective covers.** Recall that if  $A$  is an associative algebra and  $V$  is an  $A$ -module, then a *projective cover* of  $V$  is a pair  $(P, f)$  such that  $P$  is a projective  $A$ -module and  $f : P \rightarrow V$  is a superfluous epimorphism of  $A$ -modules. This means that  $f(P) = V$  and  $f(P') \neq V$  for all proper submodules  $P'$  of  $P$ . We often omit the homomorphism  $f$  and simply call  $P$  a projective cover of  $V$ .

**Definition 2.4.** For  $i \in Q_0$ , let  $p^i = \mathcal{P}e_i$ .

**Lemma 2.5.** *Assume  $Q$  is a quiver of finite type. For  $i \in Q_0$ ,  $\{p^i\}_{i \in Q_0}$  is a set of representatives of the isomorphism classes of indecomposable projective  $\mathcal{P}$ -modules. Furthermore,  $p^i$  is a projective cover of  $s^i$ .*

*Proof.* The first assertion is proved in [18, Proposition 1.9]. The fact that  $p^i$  is a projective cover of  $s^i$  follows easily.  $\square$

**Lemma 2.6.** *Assume  $Q$  is a quiver of affine (tame) or indefinite (wild) type. Then there exist  $i \in Q_0$  for which the simple module  $s^i$  does not have a projective cover.*

*Proof.* Since the module  $s^i$  is obviously cyclic, by [1, Lemma 27.3] it has a projective cover if and only if  $s^i \cong \mathcal{P}e/Ie$  for some idempotent  $e \in \mathcal{P}$  and some left ideal  $I$  contained in the Jacobson radical of  $\mathcal{P}$ . Assume this is true for some idempotent  $e$  and ideal  $I$ . Then we must have  $e = e_i$  and then  $I$  would have to contain  $\mathcal{P}_{\geq 1}e_i$ , the ideal consisting of all paths of length at least one starting at vertex  $i$ . We identify  $\mathbb{Z}Q_0$  with the root lattice via  $\sum v_j j \leftrightarrow \sum v_j \alpha_j$ . Let  $\beta$  be a minimal positive imaginary root and let  $i$  be in the support of  $\beta$  (i.e.  $\beta = \sum \beta_j \alpha_j$  with  $\beta_i > 0$ ). By [8, Theorem 1.2], there is a simple module  $T$  of  $\mathcal{P}$  whose dimension vector is  $\beta$  and so, in particular,  $\dim T_i \neq 0$ . Since the simple module  $T$  cannot be killed by  $\mathcal{P}_{\geq 1}e_i$

(since then  $T_i$  would be a proper submodule),  $\mathcal{P}_{\geq 1}e_i$  is not contained in the Jacobson radical of  $\mathcal{P}$ . This contradicts the fact that  $I$  is contained in the Jacobson radical.  $\square$

**2.4. Injective hulls.** Recall that if  $A$  is an associative algebra and  $V$  is an  $A$ -module, then an *injective hull* of  $V$  is an injective  $A$ -module  $E$  that is an essential extension of  $V$  (that is,  $V$  is a submodule of  $E$  and any nonzero submodule of  $E$  intersects  $V$  nontrivially). By the Eckmann-Schöpf Theorem, the category  $\mathcal{P}\text{-Mod}$  has enough injectives. In particular, the simple modules  $s^i$  have injective hulls. Here we give an explicit description of these injective hulls in the finite type case, and study some of their properties in the more general case

**Definition 2.7.** Assume  $Q$  is a quiver of finite type. For  $i \in Q_0$ , let  $q^i = \text{Hom}_{\mathbb{C}}(e_i\mathcal{P}, \mathbb{C})$  be the dual space of the right  $\mathcal{P}$ -module  $e_i\mathcal{P}$ . Define a left  $\mathcal{P}$ -module structure on  $q^i$  by setting  $a \cdot f(x) = f(xa)$ , for  $a \in \mathcal{P}$ ,  $f \in q^i$ , and  $x \in e_i\mathcal{P}$ .

**Lemma 2.8.** *If  $Q$  is a quiver of finite type, then  $\{q^i\}_{i \in Q_0}$  is a set of representatives of the isomorphism classes of indecomposable injective  $\mathcal{P}$ -modules. Furthermore,  $q^i$  is an injective hull of  $s^i$ .*

*Proof.* Since  $\mathcal{P} = \bigoplus_{i \in Q_0} e_i\mathcal{P}$  as right  $\mathcal{P}$ -modules, each  $e_i\mathcal{P}$  is a projective right  $\mathcal{P}$ -module. Therefore,  $q^i$  is an injective left  $\mathcal{P}$ -module (see, for instance, [15, Corollary 3.6C]). We see that  $s^i$  is (isomorphic to) the submodule of  $q^i$  spanned by the function that takes the value 1 on the path  $e_i$  and 0 on all paths of positive length. Thus, it suffices to prove that  $s^i$  is an essential submodule of  $q^i$ . Since  $Q$  is a quiver of finite type,  $\mathcal{P}$ , and hence  $q^i$ , is finite-dimensional. Choose  $0 \neq f \in \mathcal{P}$  and let  $l$  be the minimum element of  $\mathbb{N}$  such that  $f(\beta) = 0$  for all paths  $\beta$  of length greater than  $l$ . Now fix a path  $\beta = a_l \cdots a_1$  of length  $l$  such that  $f(\beta) \neq 0$ . Then  $\beta \cdot f(e_i) \neq 0$  and  $\beta \cdot f$  is zero on all paths of positive length. It follows that  $s^i$  is an essential submodule of  $q^i$ .  $\square$

For  $w = \sum_i w_i i \in \mathbb{N}Q_0$ , define the semi-simple  $\mathcal{P}$ -module

$$s^w = \bigoplus_{i \in Q_0} (s^i)^{\oplus w_i}.$$

Let  $q^i$  be the injective hull of  $s^i$  in the category  $\mathcal{P}\text{-Mod}$  (if  $Q$  is a quiver of finite type, this agrees with the notation of Definition 2.7). Then

$$q^w = \bigoplus_{i \in I} (q^i)^{\oplus w_i}$$

is the injective hull of  $s^w$ .

**Lemma 2.9.** *For  $w \in \mathbb{N}Q_0$ , any finite-dimensional submodule of  $q^w$  is nilpotent.*

*Proof.* Let  $V$  be a finite-dimensional submodule of  $q^w$ . Then we have the chain of submodules

$$V = \mathcal{P}_{\geq 0}V \supseteq \mathcal{P}_{\geq 1}V \supseteq \mathcal{P}_{\geq 2}V \supseteq \dots$$

Since  $q^w$  is an essential extension of  $s^w$ , we have that  $s^w \cap \mathcal{P}_{\geq n}V \neq 0$  for all  $n \in \mathbb{N}$  such that  $\mathcal{P}_{\geq n}V \neq 0$ . Because  $\mathcal{P}_1$  acts trivially on  $s^w$ , we have  $\dim \mathcal{P}_{\geq n+1}V < \dim \mathcal{P}_{\geq n}V$  for all  $n \in \mathbb{N}$  such that  $\mathcal{P}_{\geq n}V \neq 0$ . Thus  $\mathcal{P}_{\geq n}V = 0$  for sufficiently large  $n$ .  $\square$

**Remark 2.10.** It follows from Lemma 2.9 and Proposition 7.10 that if  $Q$  is a quiver of finite type, then  $p^w$  (and  $q^w$ ) is nilpotent. However, in general the  $p^w$  are not nilpotent.



**Proposition 2.11.** *If  $Q$  is of affine (tame) type, then  $q^w$  is locally nilpotent for all  $w \in \mathbb{Z}Q_0$ . If  $Q$  is of indefinite (wild) type, then  $q^w$  is not locally nilpotent for any  $w \in \mathbb{Z}Q_0$ .*

The following proof was explained to us by W. Crawley-Boevey.

*Proof.* It suffices to consider the case where  $w = i$  for some  $i \in Q_0$  and  $Q$  is connected. We identify  $\mathbb{Z}Q_0$  with the root lattice via  $\sum v_i i \leftrightarrow \sum v_i \alpha_i$ . We first assume that  $Q$  is of wild type. Let  $\beta$  be a minimal positive imaginary root. Thus  $(\beta, i) \leq 0$  for all  $i \in Q_0$ . Suppose the support of  $\beta$  is all of  $Q_0$ . Since  $Q$  is wild,  $\beta$  cannot be a radical vector (see [14, Theorem 4.3]), so  $(\beta, i) < 0$  for some  $i \in Q_0$ . If, on the other hand, the support of  $\beta$  is not all of  $Q_0$ , we take  $i \in Q_0$  to be a vertex not in the support of  $\beta$  but connected to it by an arrow and we again have  $(\beta, i) < 0$ . By [8, Theorem 1.2], there is a simple module  $T$  for the preprojective algebra of dimension  $\beta$ . By [7, Lemma 1],  $\text{Ext}^1(T, s^i)$  is nonzero. Let  $V$  be a nontrivial extension of  $T$  by  $s^i$ . This module must imbed in the injective hull  $q^i$  of  $s^i$  and thus  $q^i$  cannot be locally nilpotent.

Now assume that  $Q$  is of tame type. Since the preprojective algebra of a tame quiver is a finitely generated  $\mathbb{C}$ -algebra, noetherian, and a polynomial identity ring [2, Theorem 6.5] (see [25] for a proof that the preprojective algebra considered there is the same as the one considered here), any simple module is finite-dimensional (see [19, Theorem 13.10.3]). By [13, Theorem 2], the injective hull of a simple  $\mathcal{P}$ -module is artinian. In particular, finitely generated submodules of injective hulls of simple modules are artinian and noetherian. Thus they are of finite length and hence finite-dimensional. Now, the dimension vectors of simple  $\mathcal{P}$ -modules are the coordinate vectors  $i \in Q_0$  and the minimal imaginary root  $\delta$ . Since  $(\delta, i) = 0$  for all  $i \in Q_0$ , there are no nontrivial extensions between simples of dimension  $\delta$  and the one-dimensional simples. Therefore, the composition factors of the finite-dimensional submodules of the injective hull  $q^i$  of  $s^i$  are all one-dimensional simple modules. Thus  $q^i$  is locally nilpotent.  $\square$

**Remark 2.12.** In types  $A$  and  $D$ , there exist simple and explicit descriptions of the representations  $q^i$ ,  $i \in Q_0$ , in terms of classical combinatorial objects such as Young diagrams (see [11, 28, 29]). This allows one to give simple and explicit descriptions of the injective modules  $q^w$  for any  $w \in \mathbb{N}Q_0$  when the underlying graph of the corresponding quiver is of type  $A$  or  $D$ .

## 2.5. Quiver grassmannians.

**Definition 2.13** (Quiver grassmannian). For a  $\mathbb{C}Q$ -module  $V$ , let  $\text{Gr}_Q(V)$  be the variety of all  $\mathbb{C}Q$ -submodules of  $V$ . We have a natural decomposition

$$\text{Gr}_Q(V) = \bigsqcup_{u \in \mathbb{N}Q_0} \text{Gr}_Q(u, V), \quad \text{Gr}_Q(u, V) = \{U \in \text{Gr}_Q(V) \mid \dim U = u\}.$$

We call  $\text{Gr}_Q(u, V)$  a *quiver grassmannian*. Note that  $\text{Gr}_Q(u, V)$  is a closed subset of the usual grassmannian of dimension  $u$  subspaces of  $V$  and thus is a projective variety. If  $V$  is a  $\mathcal{P}$ -module, then  $\mathcal{P}$ -submodules of  $V$  are the same as  $\mathbb{C}\hat{Q}$ -submodules of  $V$ . Hence one can think of  $\text{Gr}_{\hat{Q}}(V)$  as the variety of all  $\mathcal{P}$ -submodules of  $V$ . Therefore, we will often write  $\text{Gr}_{\mathcal{P}}(V)$  and  $\text{Gr}_{\mathcal{P}}(u, V)$  for  $\text{Gr}_{\hat{Q}}(V)$  and  $\text{Gr}_{\hat{Q}}(u, V)$  when  $V$  is a  $\mathcal{P}$ -module.

**Example 2.14** (Grassmannians). If  $Q$  is the quiver with a single vertex and no arrows, then  $\mathcal{P} = \mathbb{C}$  and  $\mathcal{P}$ -modules are simply vector spaces. Then  $\text{Gr}_{\mathcal{P}}(u, V) = \text{Gr}(u, V)$  is the usual grassmannian of dimension  $u$  subspaces of  $V$ .

**Definition 2.15.** For  $V \in \mathcal{P}\text{-Mod}$ , we define a natural action of  $\text{Aut}_{\mathcal{P}} V$  on  $\text{Gr}_{\mathcal{P}}(u, V)$  given by

$$(g, U) \mapsto g(U), \quad g \in \text{Aut}_{\mathcal{P}} V, \quad U \in \text{Gr}_{\mathcal{P}}(u, V).$$

### 3. QUIVER VARIETIES

In this section we briefly recall the quiver varieties of Lusztig and Nakajima, referring the reader to [16, 20, 21] for further details, as well as the Demazure quiver varieties introduced by the first author in [30]. We fix a quiver  $Q = (Q_0, Q_1)$  and let  $\mathcal{P} = \mathcal{P}(Q)$  denote its preprojective algebra.

**3.1. Lusztig and Nakajima quiver varieties.** For  $V \in \mathcal{P}_0\text{-mod}$ , define

$$\text{End}_{\tilde{Q}} V = \bigoplus_{a \in \tilde{Q}_1} \text{Hom}_{\mathbb{C}}(V_{s(a), t(a)}).$$

For a path  $\beta = a_l \cdots a_1$  in  $Q$  and  $x = (x_a)_{a \in \tilde{Q}_1} \in \text{End}_{\tilde{Q}} V$ , we define  $x_{\beta} = x_{a_l} \cdots x_{a_1}$ . For an element  $\sum_j c_j \beta_j \in \mathbb{C}Q$ , we define

$$x_{\sum_j c_j \beta_j} = \sum_j c_j x_{\beta_j}.$$

Thus each  $x \in \text{End}_{\tilde{Q}} V$  defines a representation  $\mathbb{C}Q \rightarrow \text{End}_{\mathbb{C}} V$  of graded dimension  $\dim_{Q_0} V$  (i.e. whose induced representation of  $(\mathbb{C}Q)_0$  is in the isomorphism class determined by  $\dim_{Q_0} V$ ). Furthermore, each such representation comes from an element of  $x \in \text{End}_{\tilde{Q}}$ . These two statements are simply the equivalence of categories between the representations of the quiver and of the path algebra. We say that  $x$  is *nilpotent* if there exists  $N > 0$  such that  $x_{\beta} = 0$  for all paths  $\beta$  of length greater than  $N$ .

**Definition 3.1** (Lusztig quiver variety). For  $V \in \mathcal{P}_0\text{-mod}$ , define  $\Lambda(V) = \Lambda_Q(V)$  to be the set of all nilpotent  $\mathcal{P}$ -module structures on  $V$  compatible with its  $\mathcal{P}_0$ -module structure. More precisely,

$$\Lambda(V) = \left\{ x \in \text{End}_{\tilde{Q}} V \left| \sum_{a \in Q_1, t(a)=i} x_a x_{\bar{a}} - \sum_{a \in Q_1, s(a)=i} x_{\bar{a}} x_a = 0 \quad \forall i \in Q_0, \quad x \text{ nilpotent} \right. \right\}.$$

We call  $\Lambda(V)$  a *Lusztig quiver variety*.

As above, elements of  $\Lambda(V)$  are in natural one-to-one correspondence with nilpotent representations  $\mathcal{P} \rightarrow \text{End}_{\mathbb{C}} V$  of graded dimension  $\dim_{Q_0} V$ .

For  $V, W \in \mathcal{P}_0\text{-mod}$ , let  $\Lambda(V, W) = \Lambda(V) \times \text{Hom}_{\mathcal{P}_0}(V, W)$ . We say that  $(x, t) \in \Lambda(V, W)$  is *stable* if there exists no non-trivial  $x$ -invariant  $\mathcal{P}_0$ -submodule of  $V$  contained in  $\ker t$ . This is equivalent to the condition that  $\ker((x, t)|_{V_i}) = 0$  for all  $i \in Q_0$  (see [11, Lemma 3.4] – while the statement there is for type  $A$ , the proof carries over to the more general case). We denote the set of stable elements by  $\Lambda(V, W)^{\text{st}}$ . There is a natural action of  $\text{Aut}_{\mathcal{P}_0} V$  on  $\Lambda(V, W)$



and the restriction to  $\Lambda(V, W)^{\text{st}}$  is free (see [20, 21]). We denote the  $\text{Aut}_{\mathcal{P}_0} V$ -orbit through a point  $(x, t)$  by  $[x, t]$ .

**Definition 3.2** (Lagrangian Nakajima quiver variety). For  $V, W \in \mathcal{P}_0\text{-mod}$ , let  $\mathfrak{L}(V, W) = \Lambda(V, W)^{\text{st}} / \text{Aut}_{\mathcal{P}_0} V$ . We call  $\mathfrak{L}(V, W)$  a *lagrangian Nakajima quiver variety*. Up to isomorphism, this variety depends only on  $v = \dim_{Q_0} V$  and  $w = \dim_{Q_0} W$  and so we will sometimes denote it by  $\mathfrak{L}(v, w)$ .

**Remark 3.3.** The quiver varieties defined above are lagrangian subvarieties of what are usually called the Nakajima quiver varieties [20, 21].

**3.2. Group actions.** Let  $G_w = \text{Aut}_{\mathcal{P}_0} W = \prod_{i \in Q_0} GL(W_i)$  and let  $G_{\mathcal{P}}$  be the group of algebra automorphisms of  $\mathcal{P}$  that fix  $\mathcal{P}_0$ . The group  $G_w$  acts naturally on  $\text{Hom}_{\mathcal{P}_0}(V, W)$ . As above, we identify elements of  $\Lambda(V)$  with nilpotent representations  $\mathcal{P} \rightarrow \text{End}_{\mathbb{C}} V$  of graded dimension  $\dim_{Q_0} V$ . Then

$$(h, (x, t)) \mapsto (h \star x, t), \quad h \star x = x \circ h^{-1}, \quad h \in G_{\mathcal{P}},$$

defines a  $G_{\mathcal{P}}$ -action on  $\Lambda(V, W)$ . The actions of  $G_w$  and  $G_{\mathcal{P}}$  commute and both commute with the  $\text{Aut}_{\mathcal{P}_0} V$ -action. Since they also preserve the stability condition, they define a  $G_w \times G_{\mathcal{P}}$ -action on  $\mathfrak{L}(v, w)$ .

We can use this action to define  $G_w \times \mathbb{C}^*$ -actions on  $\mathfrak{L}(v, w)$  as follows. Suppose a function  $m : Q_1 \rightarrow \mathbb{Z}$  is given such that  $m(a) = -m(\bar{a})$  for all  $a \in Q_1$ . Then the map  $a \mapsto z^{m(a)+1}a$ ,  $z \in \mathbb{C}^*$ , extends to an automorphism of  $\mathcal{P}$  fixing  $\mathcal{P}_0$ . We denote this automorphism by  $h_m(z)$ . Thus  $h_m$  defines a group homomorphism  $\mathbb{C}^* \rightarrow G_{\mathcal{P}}$ . Then the homomorphism

$$(3.1) \quad G_w \times \mathbb{C}^* \rightarrow G_w \times G_{\mathcal{P}}, \quad (g, z) \mapsto (zg, h_m(z))$$

defines a  $G_w \times \mathbb{C}^*$ -action on  $\mathfrak{L}(v, w)$  which we denote by  $\star_m$ .

We give two important examples of this action (see [22, §2.7] and [23]). First, for each pair  $i, j \in Q_0$  connected by at least one edge, let  $b_{ij}$  denote the number of arrows in  $Q_1$  joining  $i$  and  $j$ . We fix a numbering  $a_1, \dots, a_{b_{ij}}$  of these arrows, which induces a numbering  $\bar{a}_1, \dots, \bar{a}_{b_{ij}}$  of the corresponding arrows in  $\bar{Q}_1$ . Define  $m_1 : H \rightarrow \mathbb{Z}$  by

$$m_1(a_p) = b_{ij} + 1 - 2p, \quad m_1(\bar{a}_p) = -b_{ij} - 1 + 2p.$$

For the second action, we define  $m_2(a) = 0$  for all  $a \in Q_1$ .

**3.3. Demazure quiver varieties.** Let  $\mathfrak{g}$  be the Kac-Moody algebra corresponding to the underlying graph of  $Q$  (i.e. whose Dynkin diagram is this graph) and let  $\mathcal{W}$  be its Weyl group. Recall that  $\mathcal{W}$  acts naturally on the weight lattice of  $\mathfrak{g}$ . For  $u \in \mathbb{Z}Q_0$ , we define elements of the weight and root lattice by

$$\omega_u = \sum_{i \in Q_0} u_i \omega_i, \quad \alpha_u = \sum_{i \in Q_0} u_i \alpha_i.$$

**Proposition/Definition 3.4** ([30, Proposition 5.1]). The lagrangian Nakajima quiver variety  $\mathfrak{L}(v, w)$  is a point if and only if  $\omega_w - \alpha_v = \sigma(\omega_w)$  for some  $\sigma \in \mathcal{W}$  (i.e.  $\omega_w - \alpha_v$  is an extremal weight). In this case, we let  $(x^{w, \sigma}, t^{w, \sigma})$  be a representative (unique up to isomorphism) of the  $\text{Aut}_{\mathcal{P}_0} V$ -orbit corresponding to this point. So  $\mathfrak{L}(v, w) = \{[x^{w, \sigma}, t^{w, \sigma}]\}$ .

**Definition 3.5** (Demazure quiver variety). For  $\sigma \in W$  and  $v, w \in \mathbb{N}Q_0$ , let  $\mathfrak{L}_\sigma(v, w)$  be the subvariety consisting of all  $[x, t] \in \mathfrak{L}(v, w)$  such that  $(x, t)$  is isomorphic to a subrepresentation of  $(x^{w, \sigma}, t^{w, \sigma})$ . We call  $\mathfrak{L}_\sigma(v, w)$  a *Demazure quiver variety*.

**Remark 3.6.** It follows from the uniqueness statement in Proposition/Definition 3.4 that the  $G_w \times G_{\mathcal{P}}$ -action on  $\mathfrak{L}(v, w)$  fixes  $\mathfrak{L}_\sigma(v, w)$  for all  $\sigma \in \mathcal{W}$ . Thus we have an induced  $G_w \times G_{\mathcal{P}}$ -action on the Demazure quiver varieties.

#### 4. RELATION BETWEEN QUIVER GRASSMANNIANS AND QUIVER VARIETIES

**4.1. Lagrangian Nakajima quiver varieties as quiver grassmannians.** In this section we show that certain quiver grassmannians are homeomorphic to the lagrangian Nakajima quiver varieties. We begin with a key technical proposition.

**Proposition 4.1.** *Suppose  $A = \bigoplus_{n \geq 0} A_n$  is a graded algebra and  $V$  is a locally nilpotent  $A$ -module. Furthermore, suppose  $S$  is a semisimple  $A$ -module with injective hull  $E$ .*

- (i) *Let  $\pi : E \rightarrow S$  be a projection of  $A_0$ -modules and let  $\tau : V \rightarrow S$  be a homomorphism of  $A_0$ -modules. Then there exists a unique  $A$ -module homomorphism  $\gamma : V \rightarrow E$  such that the following diagram commutes:*

$$\begin{array}{ccc} & & E \\ & \nearrow \gamma & \downarrow \pi \\ V & \xrightarrow{\tau} & S \end{array}$$

*Furthermore, the map  $\gamma$  is injective if and only if  $\tau|_{\text{socle } V}$  is injective.*

- (ii) *Suppose  $\pi_1, \pi_2 : E \rightarrow S$  are projections of  $A_0$ -modules. Then there exists a unique  $\gamma \in \text{Aut}_{\mathcal{P}} E$  such that  $\pi_2 = \pi_1 \gamma$ . The map  $\gamma$  fixes  $S$  pointwise. Conversely, given an  $A_0$ -module projection  $\pi : E \rightarrow S$  and any  $\gamma \in \text{Aut}_{\mathcal{P}} E$  fixing  $S$  pointwise,  $\pi \gamma : E \rightarrow S$  is also an  $A_0$ -module projection.*

*Proof.* Let  $V'$  be the unique maximal submodule of  $V$  such that  $\tau|_{\text{socle } V'} = 0$ . Then we must have  $\gamma|_{V'} = 0$  and so  $\gamma$  and  $\tau$  factor through  $V/V'$ . Thus, in proving part (i), it suffices to assume that  $\tau|_{\text{socle } V}$  is injective. Since  $V$  is locally nilpotent,  $\gamma(V)$  must be contained in the injective hull of  $\tau(V) \subseteq S$ . Therefore we may assume that the restriction of  $\tau$  to  $\text{socle } V$  is an isomorphism. Also, since  $V$  is locally nilpotent, we have a filtration

$$0 = V^{(0)} \subseteq V^{(1)} = \text{socle } V \subseteq V^{(2)} \subseteq V^{(3)} \subseteq \dots$$

of  $V$  where  $V^{(n)} = \{m \in V \mid A_{\geq n} \cdot m = 0\}$ . We prove by induction on  $n$  that there exists a unique homomorphism  $\gamma_n : V^{(n)} \rightarrow E$  such that the diagram

$$(4.1) \quad \begin{array}{ccc} & & E \\ & \nearrow \gamma_n & \downarrow \pi \\ V^{(n)} & \xrightarrow{\tau_n} & S \end{array}$$

commutes, where  $\tau_n = \tau|_{V^{(n)}}$ . Since  $V^{(1)} = \text{socle } V$  and  $A_+ \cdot \text{socle } V = 0$  we must have  $\gamma_1(V^{(1)}) \subseteq S$  and so the unique choice for  $\gamma_1$  is  $\tau$ . Suppose the statement holds for  $n = k$ .

Since  $E$  is injective, there exists an  $A$ -module homomorphism  $\hat{\gamma}_{k+1}$  such that the following diagram commutes:

$$\begin{array}{ccc} V^{(k+1)} & \xrightarrow{\hat{\gamma}_{k+1}} & E \\ \uparrow & \nearrow \gamma_k & \\ V^{(k)} & & \end{array}$$

Now  $V^{(k+1)} = \text{socle } V \oplus \ker \tau_{k+1}$  as vector spaces. Define  $\gamma_{k+1}$  by

$$\gamma_{k+1}(m) = \begin{cases} \tau_{k+1}(m) & \text{if } m \in \text{socle } V, \\ \hat{\gamma}_{k+1}(m) - \pi \circ \hat{\gamma}_{k+1}(m) & \text{if } m \in \ker \tau_{k+1}, \end{cases}$$

and extending by linearity. It is then clear that the diagram (4.1) commutes (with  $n = k + 1$ ). Note also that  $\gamma_{k+1}|_{V^{(k)}} = \gamma_k$ . We claim that  $\gamma_{k+1}$  is a homomorphism of  $A$ -modules. Since it is an  $A_0$ -module homomorphism by definition, it suffices to show it commutes with the action of  $A_+$ .

For  $r \in A_+$  and  $m \in V^{(k+1)}$ , we have  $r \cdot m \in V^{(k)}$ . Also,  $A_+ \cdot S = 0$ . Write  $m = m_1 + m_2$  where  $m_1 \in \text{socle } V$  and  $m_2 \in \ker \tau_{k+1}$ . Then

$$\begin{aligned} r \cdot \gamma_{k+1}(m) &= r \cdot (\tau_{k+1}(m_1) + \hat{\gamma}_{k+1}(m_2) - \pi \circ \hat{\gamma}_{k+1}(m_2)) \\ &= r \cdot \hat{\gamma}_{k+1}(m_2) = \hat{\gamma}_{k+1}(r \cdot m_2) = \gamma_k(r \cdot m_2) \\ &= \gamma_{k+1}(r \cdot m_2) = \gamma_{k+1}(r \cdot m_1 + r \cdot m_2) = \gamma_{k+1}(r \cdot m) \end{aligned}$$

as desired.

Now suppose that  $\gamma'_{k+1}$  is another  $\mathcal{P}$ -module homomorphism making (4.1) commute (with  $n = k + 1$ ). By the inductive hypothesis, we have  $\gamma_{k+1}|_{V^{(k)}} = \gamma'_{k+1}|_{V^{(k)}}$ . For all  $r \in A_{\geq 1}$  and  $m \in V^{(k+1)}$ , we have

$$r \cdot \gamma_{k+1}(m) = \gamma_{k+1}(r \cdot m) = \gamma'_{k+1}(r \cdot m) = r \cdot \gamma'_{k+1}(m).$$

Thus  $\gamma_{k+1}(m) - \gamma'_{k+1}(m)$  lies in  $S$ . Therefore

$$\gamma_{k+1}(m) - \gamma'_{k+1}(m) = \pi(\gamma_{k+1}(m) - \gamma'_{k+1}(m)) = \pi(\gamma_{k+1}(m)) - \pi(\gamma'_{k+1}(m)) = \tau(m) - \tau(m) = 0.$$

The induction is complete and we obtain the desired map  $\gamma$  by taking the limit.

Note that  $\gamma|_{\text{socle } V} = \tau|_{\text{socle } V}$ . Since a homomorphism of modules is injective if and only if its restriction to the socle is injective, it follows that  $\gamma$  is injective if and only if  $\tau|_{\text{socle } V}$  is injective.

We now prove (ii). By (i), there exists a unique  $A$ -module homomorphism  $\gamma : E \rightarrow E$  such that  $\pi_2 = \pi_1 \gamma$ . Similarly, there exists a unique  $A$ -module automorphism  $\tilde{\gamma} : E \rightarrow E$  such that  $\pi_1 = \pi_2 \tilde{\gamma}$  and  $\gamma \tilde{\gamma} = \tilde{\gamma} \gamma = \text{id}$  by the uniqueness assertion in (i). Thus  $\gamma$  is an  $A$ -automorphism of  $E$ .  $\square$

**Remark 4.2.** The map  $\pi : E \rightarrow S$  in Proposition 4.1 is equivalent to choosing an  $A_0$ -module decomposition  $E = S \oplus T$ . The second part of the proposition states that any two such decompositions are related by a unique  $A$ -module automorphism of  $E$  fixing  $S$ .

**Definition 4.3.** Let  $V$  be a  $\mathcal{P}_0$ -module of graded dimension  $v$ . We define  $\widehat{\text{Gr}}(v, q^w)$  to be the variety of injective  $\mathcal{P}_0$ -module homomorphisms  $\gamma : V \rightarrow q^w$  whose image is a  $\mathcal{P}$ -submodule of  $q^w$ .

**Theorem 4.4.** *Fix  $v, w \in \mathbb{N}Q_0$ . Then there is a bijective  $\text{Aut}_{\mathcal{P}_0} V$ -equivariant algebraic map from  $\widehat{\text{Gr}}_{\mathcal{P}}(v, q^w)$  to  $\Lambda(v, w)^{\text{st}}$  and a bijective algebraic map from  $\text{Gr}_{\mathcal{P}}(v, q^w)$  to  $\mathfrak{L}(v, w)$ . In particular,  $\widehat{\text{Gr}}_{\mathcal{P}}(v, q^w)$  is homeomorphic to  $\Lambda(v, w)^{\text{st}}$  and  $\text{Gr}_{\mathcal{P}}(v, q^w)$  is homeomorphic to  $\mathfrak{L}(v, w)$ .*

*Proof.* Fix  $V \in \mathcal{P}_0\text{-mod}$  of graded dimension  $v$  and a  $\mathcal{P}_0$ -module homomorphism  $\pi : q^w \rightarrow s^w$  that is the identity on  $s^w$ . We identify  $s^w$  with the  $W$  appearing in the definition of the quiver varieties. A point  $\gamma \in \widehat{\text{Gr}}_{\mathcal{P}}(v, q^w)$  defines an embedding of  $V$  into  $q^w$ , hence a  $\mathcal{P}$ -module structure on  $V$  satisfying the stability condition and so a point of  $\Lambda(v, w)^{\text{st}}$ . More precisely,  $\gamma \in \widehat{\text{Gr}}_{\mathcal{P}}(v, q^w)$  corresponds to the point  $(\gamma^{-1}x^w\gamma, \pi\gamma) \in \Lambda(v, w)^{\text{st}}$ . Thus we have a map

$$\iota : \widehat{\text{Gr}}_{\mathcal{P}}(v, q^w) \rightarrow \Lambda(V, W)^{\text{st}},$$

which is clearly algebraic and  $\text{Aut}_{\mathcal{P}_0} V$ -equivariant. By Proposition 4.1,  $\iota$  is bijective. Passing to the quotient by  $\text{Aut}_{\mathcal{P}_0} V$  we also obtain a bijective algebraic map  $\bar{\iota}$  from  $\text{Gr}_{\mathcal{P}}(v, q^w)$  to  $\mathfrak{L}(v, w)$ .

Now,  $\text{Gr}_{\mathcal{P}}(v, q^w)$  and  $\mathfrak{L}(v, w)$  are both projective. By, for example, [12, Theorem 4.9 and Exercise 4.4], the image of a projective variety under an algebraic map is always closed, so  $\bar{\iota}$  takes closed subsets to closed subsets. Since  $\bar{\iota}$  is a bijection, this implies that  $\bar{\iota}^{-1}$  is continuous. Hence  $\bar{\iota}$  is a homeomorphism. Since  $\widehat{\text{Gr}}_{\mathcal{P}}(v, q^w)$  and  $\Lambda(v, w)^{\text{st}}$  are principal  $G$ -bundles over  $\text{Gr}_{\mathcal{P}}(v, q^w)$  and  $\mathfrak{L}(v, w)$ , the map  $\iota$  also induces a homeomorphism.  $\square$

**Remark 4.5.** Lusztig [17, 18] has described a canonical bijection between the lagrangian Nakajima quiver varieties and grassmannian type varieties inside the projective modules  $p^w$  (see Section 7). In several places in the literature, it is claimed that the varieties defined by Lusztig are isomorphic (as algebraic varieties) to the lagrangian Nakajima quiver varieties. However, the authors are not aware of a proof existing in the literature. Most references for this statement are to Lusztig's papers [17, 18], where the points of the two varieties are shown to be in canonical bijection (similar to the situation in the current paper). Lusztig has informed the authors of the current paper that he is not aware of a proof that the varieties are isomorphic.

**Remark 4.6.**

- (i) Note that the role of the map  $\pi$  in Proposition 4.1 is to ensure the uniqueness  $\gamma$ . It follows from the proof that any map  $\gamma : V \rightarrow E$  extending the map  $\tau|_{\text{socle } V} : \text{socle } V \rightarrow S$  has the same image, regardless of its relationship with  $\pi$ .
- (ii) In the case when  $Q$  is of finite type, the injective module  $q^w$  is also projective (see Proposition 7.10) and thus Theorem 4.4 follows from [18, §2.1].
- (iii) The isomorphisms of Theorem 4.4 depend on the choice of projection  $\pi : q^w \rightarrow s^w$ . By Proposition 4.1(ii), isomorphisms coming from different projections are related by an automorphism of  $q^w$  fixing  $s^w$ .
- (iv) In Lusztig's grassmannian type realization of the lagrangian Nakajima quiver varieties [17, 18], one must require that the submodules contain paths of large enough length (this corresponds to the nilpotency condition in the definition of the quiver varieties). In the current approach using injective modules, no such condition is required due to Lemma 2.9.

**4.2. Demazure quiver grassmannians.** As before, let  $\mathfrak{g}$  be the Kac-Moody algebra corresponding to the underlying graph of  $Q$  and let  $\mathcal{W}$  be its Weyl group with Bruhat order  $\preceq$ .

**Definition 4.7.** For each  $w \in \mathbb{N}Q_0$ , we define an action of  $\mathcal{W}$  on  $\mathbb{Z}Q_0$  as follows. For  $v \in \mathbb{Z}Q_0$  and  $\sigma \in \mathcal{W}$ , define  $\sigma \cdot_w v = u$  where  $u$  is the unique element of  $\mathbb{Z}Q_0$  satisfying

$$\sigma(\omega_w - \alpha_v) = \omega_w - \alpha_u.$$

We say that  $v \in \mathbb{N}Q_0$  is  $w$ -*extremal* if  $v \in \mathcal{W} \cdot_w 0$ .

**Lemma 4.8.** *If  $v, w \in \mathbb{N}Q_0$  and  $\omega_w - \alpha_v$  is a weight of the irreducible highest weight representation of  $\mathfrak{g}$  of highest weight  $\omega_w$  (i.e the corresponding weight space is nonzero), then  $\sigma \cdot_w v \in \mathbb{N}Q_0$  for all  $\sigma \in \mathcal{W}$ . In particular  $\mathcal{W} \cdot_w 0 \subseteq \mathbb{N}Q_0$ .*

*Proof.* This follows easily from the fact that  $\mathcal{W}$  acts on the weights of highest weight irreducible representations and the weight multiplicities are invariant under this action.  $\square$

**Proposition 4.9.** *For  $v \in \mathbb{N}Q_0$ , the following statements are equivalent:*

- (i)  $v$  is  $w$ -extremal,
- (ii)  $\mathfrak{L}(v, w)$  consists of a single point,
- (iii)  $\text{Gr}_{\mathcal{P}}(v, q^w)$  consists of a single point, and
- (iv) there is a unique submodule of  $q^w$  of graded dimension  $v$ .

*Proof.* The equivalence of (i) and (ii) is given in [30, Proposition 5.1]. The equivalence of (ii), (iii) and (iv) follows from Theorem 4.4.  $\square$

**Definition 4.10** (Demazure quiver grassmannian). For  $\sigma \in \mathcal{W}$ , we let  $q^{w, \sigma}$  denote the unique submodule of  $q^w$  of graded dimension  $\sigma \cdot_w 0$ . We call  $\text{Gr}_{\mathcal{P}}(v, q^{w, \sigma})$  a *Demazure quiver grassmannian*.

**Proposition 4.11.** *If  $\sigma_1, \sigma_2 \in \mathcal{W}$  with  $\sigma_1 \preceq \sigma_2$ , then  $q^{w, \sigma_2}$  has a unique submodule of graded dimension  $\sigma_1 \cdot_w 0$  and this submodule is isomorphic to  $q^{w, \sigma_1}$ .*

*Proof.* Since  $\sigma_1 \preceq \sigma_2$ , we have  $L_{\omega_w, \sigma_1} \subseteq L_{\omega_w, \sigma_2}$ , where  $L_{\omega_w, \sigma_i}$  is the Demazure module corresponding to  $L_{\omega_w}$  (the irreducible integrable highest weight  $\mathfrak{g}$ -module with highest weight  $\omega_w$ ) and  $\sigma_i$ . It then follows from [30, Theorem 7.1] that  $q^{w, \sigma_1}$  is (isomorphic to) a submodule of  $q^{w, \sigma_2}$ . Since any submodule of  $q^{w, \sigma_2}$  is also a submodule of  $q^w$ , uniqueness follows immediately from Proposition 4.9.  $\square$

**Proposition 4.12.** *Fix  $\sigma \in \mathcal{W}$  and  $w \in \mathbb{N}Q_0$ . Then  $\text{Gr}_{\mathcal{P}}(v, q^{w, \sigma})$  is homeomorphic to the Demazure quiver variety  $\mathfrak{L}_{\sigma}(v, w)$ .*

*Proof.* This follows immediately from Definitions 3.5 and 4.10 and the description of the homomorphism  $\text{Gr}_{\mathcal{P}}(v, q^w) \cong \mathfrak{L}(v, w)$  given in Theorem 4.4.  $\square$

**Remark 4.13.** Note that if  $Q$  is a quiver of finite type and  $\sigma_0$  is the longest element of  $\mathcal{W}$ , then  $\mathfrak{L}_{\sigma_0}(v, w) = \mathfrak{L}(v, w)$  and  $\text{Gr}(v, q^{w, \sigma_0}) = \text{Gr}(v, q^w)$  for all  $v, w \in \mathbb{N}Q_0$ .

The  $(q^{w, \sigma})_{\sigma \in \mathcal{W}}$  form a directed system under the Bruhat order. Let  $\tilde{q}^w$  be the direct limit of this system.

**Lemma 4.14.** *Any locally nilpotent submodule  $V$  of  $q^w$  is contained in  $\tilde{q}^w$ .*

*Proof.* First note that for  $n \in \mathbb{N}$ , the submodule  $(q^w)^{(n)} = \{v \in q^w : \mathcal{P}_{\geq n} \cdot v = 0\}$  of  $q^w$  is finite-dimensional. This follows from the fact that  $q^i$  is a submodule of  $\text{Hom}_{\mathbb{C}}(e_i \mathcal{P}, \mathbb{C})$  (since this is an injective module containing  $s^i$ ), which has this property, and  $q^w = \bigoplus_{i \in I} (q^i)^{\oplus w_i}$ .

Since  $V$  is locally nilpotent, we have a filtration

$$0 = V^{(0)} \subseteq V^{(1)} = \text{socle } V \subseteq V^{(2)} \subseteq \dots$$

where  $V^{(n)} = \{v \in V : \mathcal{P}_{\geq n} \cdot v = 0\}$ . It suffices to show that each  $V^{(n)}$  is contained in  $\tilde{q}^w$ . Since  $V^{(n)} \subseteq (q^w)^{(n)}$ , it follows that  $V^{(n)}$  is finite-dimensional. Choose a vector space projection  $\pi : q^w \rightarrow s^w$ . By Theorem 4.4,  $V$  corresponds to a point of  $\mathfrak{L}(v, w)$ . Choose  $\sigma \in \mathcal{W}$  sufficiently large so that the  $(\omega_w - \alpha_v)$ -weight space of the representation  $L_{\omega_w}$  is contained in the Demazure module  $L_{\omega_w, \sigma}$  (we can always do this since the weight space is finite-dimensional). Then by Proposition 4.12, we have that  $V \subseteq q^{w, \sigma} \subseteq \tilde{q}^w$ .  $\square$

**Theorem 4.15.** *We have that  $\tilde{q}^w$  is the injective hull of  $s^w$  in the category  $\mathcal{P}\text{-lnMod}$ .*

*Proof.* Since each  $q^{w, \sigma}$  is nilpotent, it follows that  $\tilde{q}^w$  is locally nilpotent and thus belongs to the category  $\mathcal{P}\text{-lnMod}$ . Furthermore, it is clear that  $\tilde{q}^w$  has socle  $s^w$  and that it is an essential extension of  $s^w$ . It remains to show that  $\tilde{q}^w$  is an injective object of  $\mathcal{P}\text{-lnMod}$ . Suppose  $M$  and  $N$  are locally nilpotent  $\mathcal{P}$ -modules and we have a homomorphism  $M \rightarrow \tilde{q}^w$  and an injection  $M \hookrightarrow N$ . Since  $q^w$  is injective in the category of  $\mathcal{P}$ -modules, there exists a homomorphism  $h : N \rightarrow q^w$  such that the following diagram commutes:

$$\begin{array}{ccc} N & & \\ \uparrow & \searrow h & \\ M & \longrightarrow \tilde{q}^w \hookrightarrow & q^w \end{array}$$

Since  $N$  is locally nilpotent,  $h(N)$  is a locally nilpotent submodule of  $q^w$ . Therefore the map  $h$  factors through  $\tilde{q}^w$  by Lemma 4.14.  $\square$

**Corollary 4.16.** *We have that  $\tilde{q}^w \cong q^w$  if and only if  $Q$  is of finite or affine (tame) type.*

*Proof.* This follows immediately from Theorem 4.15 and Proposition 2.11.  $\square$

We see from the above that  $\{q^{w, \sigma}\}_{\sigma \in \mathcal{W}}$  is a “rigid” filtration of  $\tilde{q}^w$  (rigid in the sense of the uniqueness of submodules of the given  $w$ -extremal graded dimensions). Proposition 4.12 can be seen as a representation theoretic interpretation of this filtration. It corresponds to the filtration by Demazure modules of the irreducible highest-weight representation of  $\mathfrak{g}$  of highest weight  $\omega_w$ . If the quiver  $Q$  is of finite type, the Weyl group  $\mathcal{W}$ , and hence this filtration, is finite. Otherwise they are infinite. In the infinite case, we have a filtration of the infinite-dimensional  $\tilde{q}^w$  by finite-dimensional submodules  $q^{w, \sigma}$ ,  $\sigma \in \mathcal{W}$ .

## 5. GROUP ACTIONS AND GRADED QUIVER GRASSMANNIANS

In this section we define a natural  $G_w \times G_{\mathcal{P}}$ -action on the quiver grassmannians and show that the homeomorphisms of Theorem 4.4 are equivariant. We then define graded/cyclic quiver grassmannians and show that they are homeomorphic to the graded/cyclic quiver varieties of Nakajima (see [22, §4.1] and [23, §4]).



**5.1.  $G_w \times G_{\mathcal{P}}$ -action and equivariance.** Recall that  $G_w = \text{Aut}_{\mathcal{P}_0} s^w$  and  $G_{\mathcal{P}}$  is the group of algebra automorphisms of  $\mathcal{P}$  that fix  $\mathcal{P}_0$  pointwise. For a  $\mathcal{P}$ -module  $V$  and  $h \in G_{\mathcal{P}}$ , denote by  ${}^hV$  the  $\mathcal{P}$ -module with action given by  $(a, v) \mapsto h^{-1}(a) \cdot v$ . Now, fix  $(g, h) \in G_w \times G_{\mathcal{P}}$  and a  $\mathcal{P}_0$ -module homomorphism  $\pi : q^w \rightarrow s^w$ . By Proposition 4.1, there exists a unique  $\mathcal{P}$ -module homomorphism  $\gamma_{(g,h)} : {}^h q^w \rightarrow q^w$  such that the following diagram commutes:

$$\begin{array}{ccc} {}^h q^w & \xrightarrow{\gamma_{(g,h)}} & q^w \\ \pi \downarrow & & \downarrow \pi \\ s^w & \xrightarrow{g} & s^w \end{array}$$

Note that since the action of  $\mathcal{P}_0$  on  ${}^h q^w$  and  $q^w$  is the same,  $\gamma_{(g,h)}$  can be considered as a  $\mathcal{P}_0$ -automorphism of  $q^w$ . This defines a group homomorphism  $G_w \times G_{\mathcal{P}} \rightarrow \text{Aut}_{\mathcal{P}_0} q^w$ ,  $(g, h) \mapsto \gamma_{(g,h)}$ . In other words, it defines an action of  $G_w \times G_{\mathcal{P}}$  on  $q^w$  by  $\mathcal{P}_0$ -module automorphisms. This in turn defines an action on  $\widehat{\text{Gr}}_{\mathcal{P}}(v, q^w)$  and  $\text{Gr}_{\mathcal{P}}(v, q^w)$  given by

$$\begin{aligned} (g, h) \star \gamma &= \gamma_{(g,h)} \gamma, \quad \gamma \in \widehat{\text{Gr}}_{\mathcal{P}}(v, q^w) \\ (g, h) \star U &= \gamma_{(g,h)}(U), \quad U \in \text{Gr}_{\mathcal{P}}(v, q^w). \end{aligned}$$

**Proposition 5.1.** *The isomorphisms of Theorem 4.4 are  $G_w \times G_{\mathcal{P}}$ -equivariant.*

*Proof.* Let  $(x, t) \mapsto \gamma(x, t)$  be the homeomorphism  $\Lambda(v, w)^{\text{st}} \cong \widehat{\text{Gr}}_{\mathcal{P}}(v, q^w)$  of Theorem 4.4. Fix  $(x, t) \in \Lambda(v, w)^{\text{st}}$ . Recall that for  $(g, h) \in G_w \times G_{\mathcal{P}}$ , we have  $(g, h) \star (x, t) = (h \star x, gt)$ . Let  $V^x$  be the  $\mathcal{P}$ -module corresponding to  $x$ . Then  ${}^h V^x$  is the  $\mathcal{P}$ -module corresponding to  $h \star x$ . We have the commutative diagram:

$$\begin{array}{ccc} & & q^w \\ & \nearrow \gamma(x,t) & \downarrow \pi \\ V^x & \xrightarrow{t} & s^w \end{array}$$

It follows that the diagram

$$\begin{array}{ccccc} & & {}^h q^w & \xrightarrow{\gamma_{(g,h)}} & q^w \\ & \nearrow \gamma(x,t) & \downarrow \pi & & \downarrow \pi \\ {}^h V^x & \xrightarrow{t} & s^w & \xrightarrow{g} & s^w \end{array}$$

commutes. By the uniqueness statement in Proposition 4.1, we have

$$\gamma((g, h) \star (x, t)) = \gamma(h \star x, gt) = \gamma_{(g,h)} \gamma(x, t) = (g, h) \star \gamma(x, t),$$

which proves that the homeomorphism  $\Lambda(v, w)^{\text{st}} \cong \widehat{\text{Gr}}_{\mathcal{P}}(v, q^w)$  is equivariant. The remaining claim follows from the fact that the homeomorphism  $\mathfrak{L}(v, w) \cong \text{Gr}_{\mathcal{P}}(v, q^w)$  is obtained from the homeomorphism  $\Lambda(v, w)^{\text{st}} \cong \widehat{\text{Gr}}_{\mathcal{P}}(v, q^w)$  by taking quotients by  $\text{Aut}_{\mathcal{P}_0} V$ .  $\square$

**5.2. Graded/cyclic quiver grassmannians.** Fix an abelian reductive subgroup  $A$  and a group homomorphism  $\rho : A \rightarrow G_w \times G_{\mathcal{P}}$ , defining an action of  $A$  on  $q^w$  acting by  $\mathcal{P}_0$ -module automorphisms. The weight space corresponding to  $\lambda \in \text{Hom}(A, \mathbb{C}^*)$  is

$$q^w(\lambda) \stackrel{\text{def}}{=} \{v \in q^w \mid \rho(a)(v) = \lambda(a)v\}.$$

We define

$$\text{Gr}_{\mathcal{P}}(q^w)^A = \{U \in \text{Gr}_{\mathcal{P}}(q^w) \mid \rho(a) \star U = U \ \forall a \in A\}, \quad \text{Gr}_{\mathcal{P}}(u, q^w)^A = \text{Gr}_{\mathcal{P}}(q^w)^A \cap \text{Gr}_{\mathcal{P}}(u, q^w).$$

Then for all  $U \in \text{Gr}_{\mathcal{P}}(q^w)^A$ , we have the map  $\rho_U : A \rightarrow \text{Aut}_{\mathcal{P}_0} U$ ,  $a \mapsto \rho(a)|_U$ . In other words,  $\rho_U$  is a representation of  $A$  in the category of  $\mathcal{P}_0$ -modules. If  $\rho_1$  and  $\rho_2$  are two such representations, we write  $\rho_1 \cong \rho_2$  when  $\rho_1$  and  $\rho_2$  are isomorphic. That is,  $\rho_1 \cong \rho_2$  for  $\rho_i : A \rightarrow \text{Aut}_{\mathcal{P}_0} U_i$ , if there exists a  $\mathcal{P}_0$ -module isomorphism  $\xi : U_1 \rightarrow U_2$  such that  $\rho_2 = \xi \rho_1 \xi^{-1}$ , where  $\xi \rho_U \xi^{-1}$  denotes the homomorphism  $a \mapsto \xi \rho_U(a) \xi^{-1}$ . Then, for  $\rho_1 : A \rightarrow \text{Aut}_{\mathcal{P}_0} U$ ,  $U$  a  $\mathcal{P}_0$ -module, we define

$$\text{Gr}_{\mathcal{P}}(\rho_1, q^w)^A = \{U' \in \text{Gr}_{\mathcal{P}}(q^w)^A \mid \rho_{U'} \cong \rho_1\}.$$

Note that  $\text{Gr}_{\mathcal{P}}(\rho_1, q^w)^A$  depends only on the isomorphism class of  $\rho_1$ .

Recall the action of  $G_w \times G_{\mathcal{P}}$  on  $\Lambda(V, W)^{\text{st}}$  and  $\mathfrak{L}(v, w)$  described in Section 3.2. Define

$$\mathfrak{L}(w)^A = \{[x, t] \in \mathfrak{L}(v, w) \mid \rho(a) \star [x, t] = [x, t] \ \forall a \in A\}, \quad \mathfrak{L}(v, w)^A = \mathfrak{L}(w)^A \cap \mathfrak{L}(v, w).$$

Fix a point  $[x, t] \in \mathfrak{L}(v, w)^A$ . For every  $a \in A$ , there exists a unique  $\rho_1(a) \in \text{Aut}_{\mathcal{P}_0} V$  such that

$$(5.1) \quad \rho(a) \star (x, t) = \rho_1^{-1}(a) \cdot (x, t),$$

and the map  $\rho_1 : A \rightarrow \text{Aut}_{\mathcal{P}_0} V$  is a homomorphism. We let  $\mathfrak{L}(\rho_1, w)^A \subseteq \mathfrak{L}(v, w)^A$  be the set of  $A$ -fixed points  $y$  such that (5.1) holds for some representative  $(x, t)$  of  $y$ .

**Theorem 5.2.** *Let  $V$  be a  $\mathcal{P}_0$ -module and  $\rho_1 : A \rightarrow \text{Aut}_{\mathcal{P}_0} V$  a group homomorphism. Then  $\text{Gr}_{\mathcal{P}}(\rho_1, q^w)^A$  is homeomorphic to  $\mathfrak{L}(\rho_1, w)^A$ .*

*Proof.* Choose  $[x, t] \in \mathfrak{L}(\rho_1, w)^A$ . Let  $U = \gamma(x, t)(V)$  be the corresponding point of  $\text{Gr}_{\mathcal{P}}(v, q^w)^A$ . We want to show that  $\rho_1 \cong \rho_U$ . Let  $(g, h) \in A$  and consider the following commutative diagram:

$$\begin{array}{ccccc} & & hq^w & \xrightarrow{\gamma(g, h)} & q^w \\ & \nearrow \gamma(x, t) & \downarrow \pi & & \downarrow \pi \\ hV^x & \xrightarrow{t} & s^w & \xrightarrow{g} & s^w \xleftarrow{t} V \\ & & & & \nwarrow \gamma(x, t) \end{array}.$$

Then  $\rho_U(g, h) = \gamma(g, h)|_U$ . Note that  $\gamma(x, t)$  is an isomorphism when its codomain is restricted to  $U$  and we denote by  $\gamma(x, t)^{-1}$  the inverse of this restriction. We claim that  $\rho_1 = \tilde{\rho} \stackrel{\text{def}}{=} \gamma(x, t)^{-1} (\gamma(g, h)|_U) \gamma(x, t)$ . It suffices to show that

$$(h \star x, gt) = (g, h) \star (x, t) = \tilde{\rho}^{-1} \cdot (x, t) = (\tilde{\rho}^{-1} x \tilde{\rho}, t \tilde{\rho}).$$

We have

$$\begin{aligned}
 \tilde{\rho}^{-1}x &= \gamma(x, t)^{-1}(\gamma_{(g, h)}|_U)^{-1}\gamma(x, t)x \\
 &= \gamma(x, t)^{-1}(\gamma_{(g, h)}|_U)^{-1}x\gamma(x, t) \\
 &= \gamma(x, t)^{-1}(h \star x)(\gamma_{(g, z)}|_U)^{-1}\gamma(x, t) \\
 &= (h \star x)\gamma(x, t)^{-1}(\gamma_{(g, z)}|_U)^{-1}\gamma(x, t) \\
 &= (h \star x)\tilde{\rho}^{-1}
 \end{aligned}$$

and so  $\tilde{\rho}^{-1}x\tilde{\rho} = h \star x$ . Similarly,  $t\tilde{\rho} = t\gamma(x, t)^{-1}(\gamma_{(g, h)}|_U)\gamma(x, t) = gt$  and we are done.  $\square$

We now restrict to a special case of the above construction which has been studied by Nakajima. In particular, we define  $G_w \times \mathbb{C}^*$ -actions on the quiver grassmannians corresponding to the actions on quiver varieties described in Section 3.2.

For any function  $m : Q_1 \rightarrow \mathbb{Z}$  such that  $m(a) = -m(\bar{a})$  for all  $a \in Q_1$ , the group homomorphism (3.1) defines a  $G_w \times \mathbb{C}^*$ -action on  $q^w$ ,  $\widehat{\text{Gr}}_{\mathcal{P}}(v, q^w)$  and  $\text{Gr}_{\mathcal{P}}(v, q^w)$  which we again denote by  $\star_m$ . If  $A$  is any abelian reductive subgroup of  $G_w \times \mathbb{C}^*$ , we can consider the weight decompositions as above. For the remainder of this section, we fix  $m = m_2$  (see Section 3). That is,  $m(a) = 0$  for all  $a \in Q_1$ . We also write  $\star$  for  $\star_m$ . For  $x \in \mathcal{P}_n$ ,  $v \in q^w(\lambda)$  and  $(g, z) \in A$ , we have

$$\rho(g, z)(x \cdot v) = \gamma_{(zg, h_m(z))}(x \cdot v) = z^{-n}x \cdot \gamma_{(zg, h_m(z))}(v) = z^{-n}\lambda(g, z)v.$$

Thus  $\mathcal{P}_n : q^w(\lambda) \rightarrow q^w(l^{-n}\lambda)$ , where we write  $l^{-n}\lambda$  for the element  $L(-n) \otimes \lambda$  of  $\text{Hom}(A, \mathbb{C}^*)$  and  $L(-n) = \mathbb{C}$  with  $\mathbb{C}^*$ -module structure given by  $z \cdot v = z^{-n}v$ .

Now let  $(g, z)$  be a semisimple element of  $A$  and define

$$\text{Gr}_{\mathcal{P}}(q^w)^{(g, z)} = \{U \in \text{Gr}_{\mathcal{P}}(q^w) \mid (g, z) \star U = U\}, \quad \text{Gr}_{\mathcal{P}}(u, q^w)^{(g, z)} = \text{Gr}_{\mathcal{P}}(q^w)^{(g, z)} \cap \text{Gr}_{\mathcal{P}}(u, q^w).$$

The module  $q^w$  has a eigenspace decomposition with respect to the action of  $(g, z)$  given by

$$q^w = \bigoplus_{a \in \mathbb{C}^*} q^w(a), \quad q^w(a) = \{v \in q^w \mid (g, z) \star v = av\}.$$

Then  $\text{Gr}_{\mathcal{P}}(q^w)^{(g, z)}$  consists of those  $U \in \text{Gr}_{\mathcal{P}}(q^w)$  that are direct sums of subspaces of the weight spaces  $q^w(a)$ ,  $a \in \mathbb{C}^*$ . Thus, each  $U \in \text{Gr}_{\mathcal{P}}(q^w)^{(g, z)}$  inherits a weight space decomposition, or  $\mathbb{C}^*$ -grading,

$$U = \bigoplus_{a \in \mathbb{C}^*} U(a), \quad U(a) = \{v \in U \mid (g, z) \star v = av\}.$$

As above we see that  $\mathcal{P}_n : q^w(a) \rightarrow q^w(az^{-n})$  and  $\mathcal{P}_n : U(a) \rightarrow U(az^{-n})$ . We also regard  $s^w$  as an  $A$ -module via the composition

$$A \hookrightarrow G_w \times \mathbb{C}^* \xrightarrow{\text{projection}} G_w = \text{Aut}_{\mathcal{P}_0} s^w.$$

Thus  $s^w$  also inherits a  $\mathbb{C}^*$ -grading as above. For a  $Q_0 \times \mathbb{C}^*$ -graded vector space  $V = \bigoplus_{i \in Q_0, a \in \mathbb{C}^*} V_{i, a}$ , define the graded dimension (or character)

$$\text{char } V = \sum_{i \in Q_0, a \in \mathbb{C}^*} (\dim V_{i, a}) X_{i, a} \in \mathbb{N}[X_{i, a}]_{i \in Q_0, a \in \mathbb{C}^*}.$$

Recall that a  $\mathcal{P}_0$ -module is equivalent to an  $Q_0$ -graded vector space. Thus  $q^w$ ,  $s^w$ , and elements of  $\text{Gr}_{\mathcal{P}}(q^w)^{(g,z)}$  have natural  $Q_0 \times \mathbb{C}^*$ -gradings and we can consider their graded dimensions.

**Definition 5.3** (Graded/cyclic quiver grassmannian). For a graded dimension  $\mathbf{d} \in \mathbb{N}[X_{i,a}]_{i \in Q_0, a \in \mathbb{C}^*}$ , define

$$\text{Gr}_{\mathcal{P}}(\mathbf{d}, q^w)^{(g,z)} = \{U \in \text{Gr}_{\mathcal{P}}(q^w)^{(g,z)} \mid \text{char } U = \mathbf{d}\}.$$

We call  $\text{Gr}_{\mathcal{P}}(\mathbf{d}, q^w)^{(s,\varepsilon)}$  a *graded* (respectively *cyclic*) *quiver grassmannian* if  $z$  is not (respectively is) a root of unity.

**Theorem 5.4.** *Let  $V$  be a  $Q_0 \times \mathbb{C}^*$ -graded vector space. For a semisimple element  $(g, z) \in G_w \times \mathbb{C}^*$ , the graded/cyclic quiver grassmannian  $\text{Gr}_{\mathcal{P}}(\text{char } V, q^w)^{(g,z)}$  is homeomorphic to the lagrangian graded/cyclic quiver variety  $\mathfrak{L}^\bullet(V, s^w)$  defined in [23, §4], where  $s^w$  is considered as a  $Q_0 \times \mathbb{C}^*$ -graded vector space as above.*

*Proof.* This follows immediately from Proposition 5.1 since  $\mathfrak{L}^\bullet(V, W)$  is simply the set of points of  $\mathfrak{L}(V, W)$  fixed by a semisimple element  $(g, z)$  of  $\text{Aut}_{\mathcal{P}_0} s^w \times \mathbb{C}^*$ .  $\square$

**Remark 5.5.** In [23], Nakajima assumes the quiver  $Q$  is of *ADE* type. However, the definitions in [23, §4] extend naturally to the more general case.

## 6. GEOMETRIC CONSTRUCTION OF REPRESENTATIONS OF KAC-MOODY ALGEBRAS

For the remainder of this section, we fix a Kac-Moody algebra  $\mathfrak{g}$  with symmetric Cartan matrix and let  $W$  be its Weyl group. Let  $Q = (Q_0, Q_1)$  be a quiver whose underlying graph is the Dynkin graph of  $\mathfrak{g}$  and let  $\mathcal{P} = \mathcal{P}(Q)$  denote the corresponding path algebra. We also fix a  $\mathcal{P}_0$ -module projection  $\pi : q^w \rightarrow s^w$ , allowing us to identify  $\text{Gr}_{\mathcal{P}}(v, q^w)$  with  $\mathfrak{L}(v, w)$  as in Theorem 4.4.

**6.1. Constructible functions.** Recall that for a topological space  $X$ , a *constructible set* is a subset of  $X$  that is obtained from open sets by a finite number of the usual set theoretic operations (complement, union and intersection). A *constructible function* on  $X$  is a function that is a finite linear combination of characteristic functions of constructible sets. For a complex variety  $X$ , let  $M(X)$  denote the  $\mathbb{C}$ -vector space of constructible functions on  $X$  with values in  $\mathbb{C}$ . We define  $M(\emptyset) = 0$ . For a continuous map  $p : X \rightarrow X'$ , define

$$\begin{aligned} p^* : M(X') &\rightarrow M(X), & (p^* f')(x) &= f'(p(x)), & f' &\in M(X'), \\ p_! : M(X) &\rightarrow M(X'), & (p_! f)(x) &= \sum_{a \in \mathbb{Q}} a \chi(p^{-1}(x) \cap f^{-1}(a)), & f &\in M(X), \end{aligned}$$

where  $\chi$  denotes the Euler characteristic of cohomology with compact support.

**Lemma 6.1.** *Suppose  $X$  is a constructible subset of a topological space  $Y$  and let  $\iota : X \hookrightarrow Y$  be the inclusion map. Then*

- (i)  $\iota^*(f) = f|_X$  for  $f \in M(Y)$ , and
- (ii) for  $f \in M(X)$ ,  $\iota_!(f)$  is the extension of  $f$  by zero. That is  $\iota_!(f)(x) = f(x)$  for  $x \in X$  and  $\iota_!(f)(x) = 0$  for  $x \in Y \setminus X$ .

*Proof.* The proof is straightforward and will be omitted.  $\square$

**6.2. Raising and lowering operators.** Let  $V$  be a  $\mathcal{P}$ -module. For  $u, u' \in \mathbb{N}Q_0$  with  $u \leq u'$  (i.e.  $u = \sum u_i i$  and  $u' = \sum u'_i i$  where  $u_i \leq u'_i$  for all  $i \in Q_0$ ), define

$$(6.1) \quad \text{Gr}_{\mathcal{P}}(u, u', V) = \{(U, U') \in \text{Gr}_{\mathcal{P}}(u, V) \times \text{Gr}_{\mathcal{P}}(u', V) \mid U \subseteq U'\},$$

and let

$$\text{Gr}_{\mathcal{P}}(u, V) \xleftarrow{\pi_1} \text{Gr}_{\mathcal{P}}(u, u', V) \xrightarrow{\pi_2} \text{Gr}_{\mathcal{P}}(u', V)$$

be the natural projections given by  $\pi_1(U, U') = U$  and  $\pi_2(U, U') = U'$ . For each  $i \in I$ , define the following operators:

$$(6.2) \quad \begin{aligned} \hat{E}_i &: M(\text{Gr}_{\mathcal{P}}(u + i, V)) \rightarrow M(\text{Gr}_{\mathcal{P}}(u, V)), & \hat{E}_i f &= (\pi_1)_!(\pi_2^* f), \\ \hat{F}_i &: M(\text{Gr}_{\mathcal{P}}(u, V)) \rightarrow M(\text{Gr}_{\mathcal{P}}(u + i, V)), & \hat{F}_i f &= (\pi_2)_!(\pi_1^* f). \end{aligned}$$

where the maps  $\pi_1$  and  $\pi_2$  are as in (6.1) with  $u' = u + i$ .

**6.3. Compatibility with nested quiver grassmannians.** Suppose  $V_1 \subseteq V_2$  are  $\mathcal{P}$ -modules. Then we have the commutative diagram

$$\begin{array}{ccccc} \text{Gr}_{\mathcal{P}}(u, V_1) & \xleftarrow{\pi_1^1} & \text{Gr}_{\mathcal{P}}(u, u', V_1) & \xrightarrow{\pi_2^1} & \text{Gr}_{\mathcal{P}}(u', V_1) \\ \downarrow \iota_u & & \downarrow \iota_{u, u'} & & \downarrow \iota_{u'} \\ \text{Gr}_{\mathcal{P}}(u, V_2) & \xleftarrow{\pi_1^2} & \text{Gr}_{\mathcal{P}}(u, u', V_2) & \xrightarrow{\pi_2^2} & \text{Gr}_{\mathcal{P}}(u', V_2) \end{array}$$

where  $\iota_u$ ,  $\iota_{u'}$  and  $\iota_{u, u'}$  denote the canonical inclusions. Denote by  $\hat{E}_i^j$  and  $\hat{F}_i^j$ ,  $j = 1, 2$ , the operators defined in (6.2) for  $V = V_j$ .

**Proposition 6.2.** *We have*

- (i)  $\hat{E}_i^1 = \iota_u^* \circ \hat{E}_i^2 \circ (\iota_{u+i})_!$ , and
- (ii)  $\hat{F}_i^1 = \iota_{u+i}^* \circ \hat{F}_i^2 \circ (\iota_u)_!$ .

*Proof.* Let  $u' = u + i$ . By linearity, it suffices to prove the first statement for functions of the form  $1_X$  where  $X$  is a constructible subset of  $\text{Gr}_{\mathcal{P}}(u', V_1)$ . Then  $(\iota_{u'})_! 1_X = 1_X$ , where on the righthand side,  $X$  is viewed as a subset of  $\text{Gr}_{\mathcal{P}}(u', V_2)$ . We have

$$(\pi_2^2)^* \circ (\iota_{u'})_! 1_X = (\pi_2^2)^* 1_X = 1_{(\pi_2^2)^{-1}(X)},$$

and

$$(\iota_{u, u'})_! (\pi_2^1)^* 1_X = (\iota_{u, u'})_! 1_{(\pi_2^1)^{-1}(X)} = 1_{(\pi_2^1)^{-1}(X)}.$$

Since  $X \subseteq \text{Gr}_{\mathcal{P}}(u', V_1)$ , we have  $(\pi_2^2)^{-1}(X) = (\pi_2^1)^{-1}(X)$  and thus

$$(\pi_2^2)^* \circ (\iota_{u'})_! 1_X = (\iota_{u, u'})_! \circ (\pi_2^1)^* 1_X.$$

Therefore

$$\begin{aligned}
\iota_u^* \circ \hat{E}_i^2 \circ (\iota_{u'})! 1_X &= \iota_u^* \circ (\pi_1^2)! \circ (\pi_2^2)^* \circ (\iota_{u'})! 1_X \\
&= \iota_u^* \circ (\pi_1^2)! \circ (\iota_{u,u'})! \circ (\pi_2^1)^* 1_X \\
&= \iota_u^* \circ (\pi_1^2 \circ \iota_{u,u'})! \circ (\pi_2^1)^* 1_X \\
&= \iota_u^* \circ (\iota_u \circ \pi_1^1)! \circ (\pi_2^1)^* 1_X \\
&= \iota_u^* \circ (\iota_u)! \circ (\pi_1^1)! \circ (\pi_2^1)^* 1_X \\
&= (\pi_1^1)! \circ (\pi_2^1)^* 1_X \\
&= \hat{E}_i^1 1_X,
\end{aligned}$$

where the sixth equality holds since  $\iota_u^* \circ (\iota_u)!$  is the identity on  $M(\text{Gr}\mathcal{P}(u, V_1))$ .

We now prove the second statement. Again, it suffices to prove it for functions of the form  $1_X$  where  $X$  is a constructible subset of  $\text{Gr}\mathcal{P}(u, V_1)$ . Now, for  $U \in \text{Gr}\mathcal{P}(u', V_1)$ , we have

$$\begin{aligned}
\iota_{u'}^* \circ \hat{F}_i^2 \circ (\iota_u)! 1_X(U) &= \iota_{u'}^* \circ (\pi_2^2)! \circ (\pi_1^2)^* \circ (\iota_u)! 1_X(U) \\
&= \iota_{u'}^* \circ (\pi_2^2)! \circ (\pi_1^2)^* 1_X(U) \\
&= \iota_{u'}^* \circ (\pi_2^2)! \circ 1_{(\pi_1^2)^{-1}(X)}(U) \\
&= \chi((\pi_2^2)^{-1}(U) \cap (\pi_1^2)^{-1}(X)) \\
&= \chi((\pi_2^1)^{-1}(U) \cap (\pi_1^1)^{-1}(X)) \\
&= (\pi_2^1)! 1_{(\pi_1^1)^{-1}(X)}(U) \\
&= (\pi_2^1)! \circ (\pi_1^1)^* 1_X(U) \\
&= \hat{F}_i^1 1_X(U),
\end{aligned}$$

where the fifth equality holds since  $U \in \text{Gr}\mathcal{P}(u', V_1)$ . □

It follows from Proposition 4.12 that the Demazure quiver grassmannians stabilize in the following sense.

**Corollary 6.3.** *For  $u, w \in \mathbb{N}Q_0$ , there exists  $\sigma \in \mathcal{W}$ , such that  $\text{Gr}\mathcal{P}(v, q^{w, \sigma'})$  is homeomorphic to  $\mathfrak{L}(v, w)$  for all  $\sigma' \succeq \sigma$ .*

*Proof.* It follows from [30, Proposition 6.1] that there exists a  $\sigma \in \mathcal{W}$  such that  $\text{Gr}\mathcal{P}(v, q^{w, \sigma}) \cong \mathfrak{L}_\sigma(v, w) = \mathfrak{L}(v, w)$ . It follows from the same proposition that for  $\sigma' \succeq \sigma$ , we have  $\mathfrak{L}_{\sigma'}(v, w) = \mathfrak{L}(v, w)$ . The result then follows from Proposition 4.12. □

**Corollary 6.4.** *For  $v, w \in \mathbb{N}Q_0$ , let  $\sigma^{v, w} \in \mathcal{W}$  be minimal among the  $\sigma \in \mathcal{W}$  such that  $\text{Gr}\mathcal{P}(v, q^{w, \sigma})$  is homeomorphic to  $\mathfrak{L}(v, w)$ . Then  $\text{Gr}\mathcal{P}(v, q^{w, \sigma}) \cong \text{Gr}\mathcal{P}(v, q^w)$  for all  $\sigma \succeq \sigma^{v, w}$ . In particular, every submodule of the injective module  $q^w$  of graded dimension  $v$  is a submodule of  $q^{w, \sigma}$  for  $\sigma \succeq \sigma^{v, w}$ .*

**Remark 6.5.** In the case when  $\mathfrak{g}$  is of finite type, we can take  $\sigma = \sigma_0$ , where  $\sigma_0$  is the longest element of the Weyl group. Then  $\text{Gr}\mathcal{P}(v, q^w)$  is isomorphic to  $\text{Gr}\mathcal{P}(v, q^{w, \sigma_0})$  for all  $v \in \mathbb{N}Q_0$ .



**Lemma 6.6.** *Suppose  $w, v, v' \in \mathbb{N}Q_0$  with  $v \leq v'$  and  $\sigma \in \mathcal{W}$ . Then the diagram*

$$\begin{array}{ccccc} \mathrm{Gr}_{\mathcal{P}}(v, q^{w, \sigma}) & \xleftarrow{\pi_1} & \mathrm{Gr}_{\mathcal{P}}(v, v', q^{w, \sigma}) & \xrightarrow{\pi_2} & \mathrm{Gr}_{\mathcal{P}}(v', q^{w, \sigma}) \\ \downarrow & & \downarrow & & \downarrow \\ \mathrm{Gr}_{\mathcal{P}}(v, q^w) & \xleftarrow{\pi_1} & \mathrm{Gr}_{\mathcal{P}}(v, v', q^w) & \xrightarrow{\pi_2} & \mathrm{Gr}_{\mathcal{P}}(v', q^w) \end{array}$$

*commutes, where the vertical arrows are the natural inclusions. If  $\sigma \succeq \sigma^{v, w}, \sigma^{v', w}$ , then the vertical arrow are isomorphisms.*

*Proof.* This follows immediately from Corollary 6.4.  $\square$

**6.4. Quiver grassmannian realization of representations.** For each  $i \in I$ , define

$$(6.3) \quad H_i : M(\mathrm{Gr}_{\mathcal{P}}(v, q^w)) \rightarrow M(\mathrm{Gr}_{\mathcal{P}}(v, q^w)), \quad H_i f = (w - Cv)_i f,$$

where  $C$  is the Cartan matrix of  $\mathfrak{g}$ . Also, in the special case when  $V = q^w$  for some  $w$ , we denote the operators  $\hat{E}_i$  and  $\hat{F}_i$  by  $E_i$  and  $F_i$  respectively.

**Proposition 6.7.** *The operators  $E_i, F_i, H_i$  define an action of  $\mathfrak{g}$  on  $\bigoplus_u M(\mathrm{Gr}_{\mathcal{P}}(u, q^w))$ .*

*Proof.* Throughout this proof, for varieties  $X$  and  $Y$ , the notation  $X \cong Y$  means that  $X$  and  $Y$  are homeomorphic. In [20, §10], Nakajima defines the variety

$$\mathfrak{F}(v, w; i) \stackrel{\mathrm{def}}{=} \tilde{\mathfrak{F}}(v, w; i) / \mathrm{Aut}_{\mathcal{P}_0} V,$$

where

$$\tilde{\mathfrak{F}}(v, w; i) = \{(x, t, Z) \mid (x, t) \in \Lambda(V, W)^{\mathrm{st}}, Z \subseteq V, x(Z) \subseteq Z, \dim Z = v - i\}.$$

Using the homeomorphism of Theorem 4.4, we have

$$\tilde{\mathfrak{F}}(v, w; i) \cong \{(\gamma, Z) \mid \gamma \in \widehat{\mathrm{Gr}}_{\mathcal{P}}(v, q^w), Z \subseteq V, \dim Z = v - i, \mathcal{P} \cdot \gamma(Z) \subseteq \gamma(Z)\}.$$

The map

$$\begin{aligned} \{(\gamma, Z) \mid \gamma \in \widehat{\mathrm{Gr}}_{\mathcal{P}}(v, q^w), Z \subseteq V, \dim Z = v - i, \mathcal{P} \cdot \gamma(Z) \subseteq \gamma(Z)\} &\rightarrow \mathrm{Gr}_{\mathcal{P}}(v - i, v, q^w), \\ (\gamma, Z) &\mapsto (\gamma(Z), \gamma(V)) \end{aligned}$$

is a principle  $\mathrm{Aut}_{\mathcal{P}_0} V$ -bundle and thus

$$\begin{aligned} \mathfrak{F}(v, w; i) &= \tilde{\mathfrak{F}}(v, w; i) / \mathrm{Aut}_{\mathcal{P}_0} V \\ &\cong \{(\gamma, Z) \mid \gamma \in \widehat{\mathrm{Gr}}_{\mathcal{P}}(v, q^w), Z \subseteq V, \dim Z = v - i, \mathcal{P} \cdot \gamma(Z) \subseteq \gamma(Z)\} / \mathrm{Aut}_{\mathcal{P}_0} V \\ &= \mathrm{Gr}_{\mathcal{P}}(v - i, v, q^w). \end{aligned}$$

Therefore, the following diagram commutes:

$$(6.4) \quad \begin{array}{ccccc} \mathrm{Gr}_{\mathcal{P}}(v - i, q^w) & \xleftarrow{\pi_1} & \mathrm{Gr}_{\mathcal{P}}(v - i, v, q^w) & \xrightarrow{\pi_2} & \mathrm{Gr}_{\mathcal{P}}(v, q^w) \\ \downarrow \cong & & \downarrow \cong & & \downarrow \cong \\ \mathfrak{L}(v - i, w) & \xleftarrow{\pi_1} & \mathfrak{F}(v, w; i) & \xrightarrow{\pi_2} & \mathfrak{L}(v, w) \end{array}$$

where the maps  $\pi_1$  and  $\pi_2$  appearing the bottom row are described in [20, §10]. The result then follows immediately from [20, Proposition 10.12].  $\square$

Let  $U(\mathfrak{g})^-$  be the lower half of the enveloping algebra of  $\mathfrak{g}$ . Then let  $\alpha$  be the constant function on  $\text{Gr}_{\mathcal{P}}(0, q^w)$  with value 1 and let

$$(6.5) \quad L_w \stackrel{\text{def}}{=} U(\mathfrak{g})^- \cdot \alpha \subseteq \bigoplus_v M(\text{Gr}_{\mathcal{P}}(v, q^w)),$$

$$(6.6) \quad L_w(v) \stackrel{\text{def}}{=} M(\text{Gr}_{\mathcal{P}}(v, q^w)) \cap L_w.$$

**Theorem 6.8.** *The operators  $E_i, F_i, H_i$  preserve  $L_w$  and  $L_w$  is isomorphic to the irreducible highest-weight integrable representation of  $\mathfrak{g}$  with highest weight  $\omega_w$ . The summand  $L_w(v)$  in the decomposition  $L_w = \bigoplus_v L_w(v)$  is a weight space with weight  $\omega_w - \alpha_v$ .*

*Proof.* In light of the commutative diagram (6.4), the result follows immediately from [20, Theorem 10.14].  $\square$

**Remark 6.9.** By Proposition 6.2 and Lemma 6.6, we can always work with  $\text{Gr}_{\mathcal{P}}(v, q^{w, \sigma})$  for large enough  $\sigma$ . Therefore, we can avoid quiver grassmannians in infinite dimensional injectives if desired.

From the realization of irreducible highest-weight representations given in Theorem 6.8, we obtain some natural automorphisms of these representations. Recall from Definition 2.15 the natural action of  $\text{Aut}_{\mathcal{P}} q^w$  on  $\text{Gr}_{\mathcal{P}}(v, q^w)$  for any  $v$  given by  $(g, V) \mapsto g(V)$ . This induces an action on  $\bigoplus_v M(\text{Gr}_{\mathcal{P}}(v, q^w))$  given by

$$(g, f) \mapsto f \circ g^{-1}, \quad f \in \bigoplus_v M(\text{Gr}_{\mathcal{P}}(v, q^w)), \quad g \in \text{Aut}_{\mathcal{P}} q^w.$$

This action clearly commutes with the operators  $E_i$  and  $F_i$  and thus induces an action on  $L_w$ . Such actions do not seem to be clear in the original quiver variety picture. Similar actions have been considered by Lusztig [18, §1.22] in the case when  $Q$  is of finite type.

## 7. RELATION TO LUSZTIG'S GRASSMANNIAN REALIZATION

In [17, 18], Lusztig gave a grassmannian type realization of the lagrangian Nakajima quiver varieties inside the projective modules  $p^w$ . In the case when  $Q$  is a quiver of finite type, the injective hulls of the simple objects are also projective covers (of different simple objects). Thus, Lusztig's and our construction are closely related. In this section, we extend Lusztig's construction to give a realization of the Demazure quiver varieties. We then give a precise relationship between his construction and ours in the finite type case. We will see that the natural identification of the two constructions corresponds to the Chevalley involution on the level of representations of the Lie algebra  $\mathfrak{g}$  associated to our quiver.

### 7.1. Lusztig's construction and Demazure quiver varieties.

**Definition 7.1.** For  $V \in \mathcal{P}\text{-Mod}$ , define

$$\tilde{\text{Gr}}_{\mathcal{P}}(V) = \{U \in \text{Gr}_{\mathcal{P}}(V) \mid \mathcal{P}_n \cdot V \subseteq U \text{ for some } n \in \mathbb{N}\}.$$

In other words,  $\tilde{\text{Gr}}_{\mathcal{P}}(V)$  consists of all  $\mathcal{P}$ -submodules of  $V$  such that the quotient  $V/U$  is nilpotent. For  $u \in \mathbb{N}Q_0$ , we define

$$\tilde{\text{Gr}}_{\mathcal{P}}(u, V) = \{U \in \tilde{\text{Gr}}_{\mathcal{P}}(V) \mid \dim_{Q_0}(V/U) = u\}.$$

**Proposition 7.2.** *Fix  $v, w \in \mathbb{N}Q_0$ . Then  $\mathfrak{L}(v, w)$  is homeomorphic to  $\tilde{\text{Gr}}_{\mathcal{P}}(v, p^w)$ .*

*Proof.* A canonical bijection from  $\tilde{\text{Gr}}_{\mathcal{P}}(v, p^w)$  to  $\mathfrak{L}(v, w)$  is defined in [17, Theorem 2.26]. Since this map is clearly algebraic and the varieties in question are projective, it is a homeomorphism as in the proof of Theorem 4.4.  $\square$

**Proposition 7.3.** *For  $v \in \mathbb{N}Q_0$ , the following statements are equivalent:*

- (i)  $v$  is  $w$ -extremal,
- (ii)  $\mathfrak{L}(v, w)$  consists of a single point,
- (iii)  $\tilde{\text{Gr}}_{\mathcal{P}}(v, p^w)$  consists of a single point, and
- (iv) there is a unique  $\mathcal{P}$ -submodule  $V$  of  $p^w$  of codimension  $v$  such that  $p^w/V$  is nilpotent.

*Proof.* The equivalence of (i) and (ii) is given in [30, Proposition 5.1]. The equivalence of (ii) and (iii) follows from Proposition 7.2. Finally, the equivalence of (iii) and (iv) follows directly from Definition 7.1.  $\square$

**Definition 7.4.** For  $\sigma \in \mathcal{W}$ , we let  $p^{w, \sigma}$  denote the unique submodule of  $p^w$  of graded codimension  $\sigma \cdot_w 0$  and define

$$\tilde{\text{Gr}}_{Q, \sigma}(v, p^w) = \{V \in \tilde{\text{Gr}}_{\mathcal{P}}(v, p^w) \mid p^{w, \sigma} \subseteq V\}.$$

**Proposition 7.5.** *Fix  $\sigma \in \mathcal{W}$  and  $v, w \in \mathbb{N}Q_0$ . Then  $\tilde{\text{Gr}}_{Q, \sigma}(v, p^w)$  is homeomorphic to the Demazure quiver variety  $\mathfrak{L}_{\sigma}(v, w)$ .*

*Proof.* This follows immediately from Definitions 3.5 and 7.4.  $\square$

**7.2. Relation between the projective and injective constructions.** We now suppose  $Q$  is of finite type and let  $\mathfrak{g}$  be the Kac-Moody algebra whose Dynkin diagram is the underlying graph of  $Q$ . Let  $\sigma_0$  be the longest element of the Weyl group of  $\mathfrak{g}$ . There is a unique Dynkin diagram automorphism  $\theta$  such that  $-w_0(\alpha_i) = \alpha_{\theta(i)}$ . Extend  $\theta$  to an automorphism of the root lattice  $\bigoplus_{i \in Q_0} \mathbb{Z}\alpha_i$  by linearly extending the map  $\alpha_i \mapsto \alpha_{\theta(i)}$ . We also have an involution of  $\mathbb{N}Q_0$  given by  $w \mapsto \theta(w)$  where  $\theta(w)_i = w_{\theta(i)}$ .

**Definition 7.6** (Chevalley involution). The *Chevalley involution*  $\zeta$  of  $\mathfrak{g}$  is given by

$$\zeta(E_i) = F_i, \quad \zeta(F_i) = E_i, \quad \zeta(H_i) = -H_i.$$

For any representation  $V$  of  $\mathfrak{g}$ , let  ${}^{\zeta}V$  be the representation with the same underlying vector space as  $V$ , but with the action of  $\mathfrak{g}$  twisted by  $\zeta$ . More precisely, the  $\mathfrak{g}$ -action on  ${}^{\zeta}V$  is given by  $(a, v) \mapsto \zeta(a) \cdot v$ .

For a dominant weight  $\lambda$  of  $\mathfrak{g}$ , let  $L_{\lambda}$  denote the corresponding irreducible highest-weight representation and let  $v_{\lambda}$  be a highest weight vector. Recall that an isomorphism of irreducible representations is uniquely determined by the image of  $v_{\lambda}$ . The following lemma is well known.

**Lemma 7.7.** *The lowest weight of  $L_{\lambda}$  is  $\sigma_0(\lambda) = -\theta(\lambda)$ . If  $v_{-\theta(\lambda)}$  denotes a lowest weight vector, then the map  $v_{\lambda} \mapsto v_{-\theta(\lambda)}$  induces an isomorphism  ${}^{\zeta}L_{\lambda} \cong L_{\theta(\lambda)}$ .*

**Lemma 7.8.** *We have  $\dim_{Q_0} p^w = \dim_{Q_0} q^w = \sigma_0 \cdot_w 0$ .*

*Proof.* Since the lowest weight of the representation  $L(w)$  is  $\sigma_0(w)$ , the result follows immediately from Theorem 4.4 and Proposition 7.2.  $\square$

**Lemma 7.9.** *For  $w \in \mathbb{N}Q_0$ , we have  $\sigma_0 \cdot_w 0 = \sigma_0 \cdot_{\theta(w)} 0$ . Furthermore,  $\theta(\sigma_0 \cdot_w 0) = \sigma_0 \cdot_w 0$ .*

*Proof.* Let  $v = \sigma_0 \cdot_w 0$ . Then  $\alpha_v = \omega_w - \sigma_0(\omega_w) = \omega_w + \theta(\omega_w)$  and the results follow easily from the fact that  $\theta^2 = \text{Id}$ .  $\square$

**Proposition 7.10.** *If  $Q$  is a quiver of finite type and  $w \in \mathbb{N}Q_0$ , then  $p^w \cong q^{\theta(w)}$ .*

*Proof.* Since  $p^w = \bigoplus_{i \in Q_0} (p^i)^{\oplus w_i}$  and  $q^w = \bigoplus_{i \in Q_0} (q^i)^{\oplus w_i}$ , it suffices to prove the result for  $w$  equal to  $i$  for arbitrary  $i \in Q_0$ .

Let  $v = \sigma_0 \cdot_w 0 = \dim_{Q_0} p^i$ . In the geometric realization of crystals via quiver varieties [26], the point  $\tilde{\text{Gr}}_{\mathcal{P}}(v, p^w) \cong \mathfrak{L}(v, w)$  corresponds to the lowest weight element of the crystal  $B_{\omega_i}$ . The lowest weight of the representation  $L_{\omega_i}$  is  $\sigma_0(\omega_i) = -\omega_{\theta(i)}$ . Therefore, it follows from the geometric description of the crystals that  $\dim_{Q_0} \text{socle } p^i = \theta(i)$ . By Lemmas 7.8 and 7.9, we have

$$\dim_{Q_0} p^i = \sigma_0 \cdot_w 0 = \sigma_0 \cdot_{\theta(w)} 0 = \dim_{Q_0} q^{\theta(i)}.$$

Thus, by Proposition 4.9, we have  $p^i \cong q^{\theta(i)}$ .  $\square$

**Corollary 7.11.** *Suppose  $Q$  is a quiver of finite type,  $w \in \mathbb{N}Q_0$ , and  $\sigma \in \mathcal{W}$ . Then  $q^{w, \sigma} \cong p^{\theta(w), \sigma \sigma_0}$ .*

*Proof.* Let  $\tau = \sigma \sigma_0$  (and so  $\sigma = \tau \sigma_0$ ). In light of Propositions 4.9, 7.3 and 7.10 and Definitions 4.10 and 7.4, it suffices, by Proposition 7.10, to prove that the codimension of  $q^{w, \sigma}$  in  $q^w$  is  $\tau \cdot_{\theta(w)} 0$ .

Let  $y = \tau \cdot_{\theta(w)} 0$ , so that

$$\tau(\theta(w)) = \theta(w) - \alpha_y \implies \alpha_y = \theta(w) - \tau(\theta(w)).$$

Now, let

$$\begin{aligned} v = \dim_{Q_0} q^w &= \sigma_0 \cdot_w 0 \implies \sigma_0(w) = w - \alpha_v \\ u = \dim_{Q_0} q^{w, \sigma} &= \sigma \cdot_w 0 \implies \sigma(w) = w - \alpha_u. \end{aligned}$$

Therefore

$$\begin{aligned} \sum_{i \in Q_0} (v_i - u_i) \alpha_i &= -\sigma_0(w) + \sigma(w) \\ &= \theta(w) + \tau \sigma_0(w) \\ &= \theta(w) - \tau(\theta(w)), \end{aligned}$$

and so  $y = v - u$  as desired.  $\square$

**Proposition 7.12.** *If  $Q$  is a quiver of finite type, then  $\text{Gr}_{\mathcal{P}}(u, q^w) \cong \tilde{\text{Gr}}_{\mathcal{P}}((\sigma_0 \cdot_w 0) - u, p^{\theta(w)})$ .*

*Proof.* Let  $(x, V)$  be the quiver representation corresponding to the  $\mathcal{P}$ -module  $q^w$  and let  $v = \dim_{Q_0} V = \sigma_0 \cdot_w 0$ . By Proposition 7.10,  $(x, V)$  also corresponds to the  $\mathcal{P}$ -module  $p^{\theta(w)}$ . By Remark 2.10,  $\mathcal{P}_n \cdot p^w = 0$  for sufficiently large  $n$ . Therefore

$$\begin{aligned} \text{Gr}_{\mathcal{P}}(u, q^w) &= \{U \subseteq V \mid x(U) \subseteq U, \dim U = u\} \\ &= \{U \subseteq V \mid x(U) \subseteq U, \dim_{Q_0} V/U = v - u\} \\ &\cong \tilde{\text{Gr}}_{\mathcal{P}}(v - u, p^{\theta(w)}). \end{aligned}$$

$\square$

By Proposition 7.12, we have  
(7.1)

$$\mathfrak{L}(u, w) \xleftarrow[\cong]{\phi_w(u)} \mathrm{Gr}_{\mathcal{P}}(u, q^w) \cong \tilde{\mathrm{Gr}}_{\mathcal{P}}((\sigma_0 \cdot_w 0) - u, p^{\theta(w)}) \xrightarrow[\cong]{\psi_{\theta(w)}((\sigma_0 \cdot_w 0) - u)} \mathfrak{L}((\sigma_0 \cdot_w 0) - u, \theta(w))$$

where  $\phi_w(u)$  is the homeomorphism of Theorem 4.4, and  $\psi_{\theta(w)}(u)$  is the homeomorphism of Proposition 7.2. Define

$$\phi_w = (\phi_w(u))_u : \mathrm{Gr}_{\mathcal{P}}(q^w) \rightarrow \bigsqcup_u \mathfrak{L}(u, w), \quad \psi_w = (\psi_w(u))_u : \tilde{\mathrm{Gr}}_{\mathcal{P}}(p^w) \rightarrow \bigsqcup_u \mathfrak{L}(u, w).$$

**Theorem 7.13.** *The homeomorphism  $\psi_{\theta(w)} \circ \phi_w^{-1}$  induces the involution  $\zeta$ . More precisely, we have  $a \circ (\psi_{\theta(w)} \circ \phi_w^{-1})^* = (\psi_{\theta(w)} \circ \phi_w^{-1})^* \circ \zeta(a)$ ,  $a \in \mathfrak{g}$ , as operators on  $L_w$ , where  $(\psi_{\theta(w)} \circ \phi_w^{-1})^*$  denotes the pullback of functions along  $\psi_{\theta(w)} \circ \phi_w^{-1}$ .*

*Proof.* For  $u, u' \in \mathbb{N}Q_0$ , define

$$\tilde{\mathrm{Gr}}_{\mathcal{P}}(u, u', p^{\theta(w)}) = \{(U, U') \in \tilde{\mathrm{Gr}}_{\mathcal{P}}(u, p^{\theta(w)}) \times \tilde{\mathrm{Gr}}_{\mathcal{P}}(u', p^{\theta(w)}) \mid U' \subseteq U\}.$$

The map  $\psi_{\theta(w)}$  induces a homeomorphism

$$\tilde{\mathrm{Gr}}_{\mathcal{P}}(u, u', p^{\theta(w)}) \xrightarrow{\cong} \mathfrak{F}(u, \theta(w); u - u')$$

for all  $u, u' \in \mathbb{N}Q_0$  and we will also denote this collection of isomorphisms by  $\psi_{\theta(w)}$ . Then we have the following commutative diagram.

$$\begin{array}{ccccc} \mathfrak{L}(u - i, w) & \xleftarrow{\pi_1} & \mathfrak{F}(u, w; i) & \xrightarrow{\pi_2} & \mathfrak{L}(u, w) \\ \uparrow \phi_w \cong & & \uparrow \phi_w \cong & & \uparrow \phi_w \cong \\ \mathrm{Gr}_{\mathcal{P}}(u - i, q^w) & \xleftarrow{\pi_1} & \mathrm{Gr}_{\mathcal{P}}(u - i, u, q^w) & \xrightarrow{\pi_2} & \mathrm{Gr}_{\mathcal{P}}(u, q^w) \\ \downarrow \cong & & \downarrow \cong & & \downarrow \cong \\ \tilde{\mathrm{Gr}}_{\mathcal{P}}((\sigma_0 \cdot_w 0) - (u - i), p^{\theta(w)}) & \xleftarrow{\pi_2} & \tilde{\mathrm{Gr}}_{\mathcal{P}}((\sigma_0 \cdot_w 0) - u, (\sigma_0 \cdot_w 0) - (u - i), p^{\theta(w)}) & \xrightarrow{\pi_1} & \tilde{\mathrm{Gr}}_{\mathcal{P}}((\sigma_0 \cdot_w 0) - u, p^{\theta(w)}) \\ \downarrow \psi_{\theta(w)} \cong & & \downarrow \psi_{\theta(w)} \cong & & \downarrow \psi_{\theta(w)} \cong \\ \mathfrak{L}((\sigma_0 \cdot_w 0) - (u - i), \theta(w)) & \xleftarrow{\pi_2} & \mathfrak{F}((\sigma_0 \cdot_w 0) - u, \theta(w); i) & \xrightarrow{\pi_1} & \mathfrak{L}((\sigma_0 \cdot_w 0) - u, \theta(w)) \end{array}$$

It follows that for  $f \in \bigoplus_u M(\mathfrak{L}(u, w))$ , we have

$$\begin{aligned} E_i \circ (\psi_{\theta(w)} \circ \phi_w^{-1})^*(f) &= (\psi_{\theta(w)} \circ \phi_w^{-1})^* \circ F_i(f), \\ F_i \circ (\psi_{\theta(w)} \circ \phi_w^{-1})^*(f) &= (\psi_{\theta(w)} \circ \phi_w^{-1})^* \circ E_i(f). \end{aligned}$$

Furthermore,  $(\psi_{\theta(w)} \circ \phi_w^{-1})^*$  maps the constant function on  $\mathfrak{L}(0, w)$  with value one to the constant function on  $\mathfrak{L}((\sigma_0 \cdot_w 0), \theta(w))$  with value one. The result follows.  $\square$

**Remark 7.14.** Note that the middle isomorphism in (7.1) depends on our identification of  $q^w$  and  $p^{\theta(w)}$ . The isomorphism  $\phi_w(u)$  also depends on our fixed projection  $\pi : q^w \rightarrow s^w$ . By Proposition 4.1, all such choices are related by the natural action of  $\mathrm{Aut}_{\mathcal{P}} q^w$  (see Definition 2.15). A similar group action appears in the identification of  $\tilde{\mathrm{Gr}}_{\mathcal{P}}((\sigma_0 \cdot_w 0) - u, p^{\theta(w)})$  with  $\mathfrak{L}((\sigma_0 \cdot_w 0) - u, \theta(w))$  (see [18]). Via the isomorphisms  $\phi_w(u)$ , the group  $\mathrm{Aut}_{\mathcal{P}} q^w$  acts

on the space of constructible functions on  $\bigsqcup_v \mathfrak{L}(v, w)$  and  $L_w$  is a subspace of the space of invariant functions. The pullback  $(\psi_{\theta(w)} \circ \phi_w^{-1})^*$  acting on the space of invariant functions is independent of the choice of  $\pi$  and the chosen identification of  $q^w$  with  $p^{\theta(w)}$ .

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